MEMBRANE HORIZONS: THE BLACK HOLE'S NEW CLOTHES

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Abstract

This thesis addresses some classical and semi-classical aspects of black holes, using an effective membrane representation of the event horizon. This "membrane paradigm" is the remarkable view that, to an external observer, a black hole appears to behave exactly like a dynamical fluid membrane, obeying such pre-relativistic equations as Ohm's law and the Navier-Stokes equation. It has traditionally been derived by manipulating the equations of motion. Here, however, the equations are derived from an underlying action formulation which has the advantage of clarifying the paradigm and simplifying the derivations, in addition to providing a bridge to thermodynamics and quantum mechanics. Within this framework, previous membrane results are derived and extended to dyonic black hole solutions. It is explained how an action can produce dissipative equations. The classical portion of the study ends with a demonstration of the validity of a minimum entropy production principle for black holes.

Turning next to semi-classical theory, it is shown that familiar thermodynamic properties of black holes also emerge from the membrane action, via a Euclidean path integral. In particular, the membrane action can account for the hole's Bekenstein-Hawking entropy, including the numerical factor. Two short and direct derivations of Hawking radiation as an instanton process are then presented. The first is a tunneling calculation based on particles in a dynamical geometry, closely analogous to Schwinger pair production in an electric field. The second derivation makes use of the membrane representation of the horizon. In either approach, the imaginary part of the action for the classically forbidden process is related to the Boltzmann factor for emission at the Hawking temperature. But because these derivations respect

conservation laws, the exact result contains a qualitatively significant correction to Hawking's thermal spectrum.

Finally, by extending the charged Vaidya metric to cover all of spacetime, a Penrose diagram for the formation and evaporation of a charged black hole is obtained. It is found that the spacetime following the evaporation of a black hole is predictable from initial conditions, provided that the dynamics of the time-like singularity can be calculated.

"Coffee ... is the fuel of science"

– John Archibald Wheeler, overheard at tea

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Notation

In this work, we use lowercase indices for four-dimensional tensors and uppercase indices for the two-dimensional tensors that occupy space-like sections of the horizon. Repeated indices are implicitly summed over.

The spacetime metric tensor is denoted by g_{ab}, the induced metric on the time-like stretched horizon is h_{ab}, the metric on a space-like slice of spacetime is ${}^3g_{ab}$, and the metric on a space-like section of the horizon is written γ_{AB}. Correspondingly, we denote the 4-covariant derivative by ∇_a, the 3-covariant derivative on the stretched horizon by $|a$, and the 2-covariant derivative on a space-like slicing by $\|A$.

We take the spacetime metric to have signature $(-+++)$. Our sign conventions are those of Misner, Thorne, and Wheeler, with the exception of the extrinsic curvature which we define to have a positive trace for a convex surface.

Throughout, we use geometrized units in which Newton's constant, G, and the speed of light, c, are set to one. We shall also usually set Planck's constant, $\hbar$, and Boltzmann's constant, k_B, to one. The appropriate factors of these constants may always be restored through dimensional analysis.

Chapter 1

Black Holes From The Outside

1.1 Causality and the Horizon

A black hole is a region of spacetime from which, crudely speaking, there is no escape. The boundary of a black hole is called an *event horizon* because an outside observer is unable to observe events on the other side of it; light rays from inside cannot propagate out, hence "black" hole.

From these basic definitions, several properties of the horizon emerge as consequences. We emphasize four that will be especially important to this study:

- The horizon is a *causal* boundary. Since no signal can get out, the inside of a black hole cannot influence the outside; we say that the black hole's interior is causally disconnected from the outside.

- The spacetime containing a solitary black hole is inherently *time-reversal asymmetric*. The time reverse of infall – escape – does not occur, so a direction for the arrow of time is implicit.

- Because nothing can travel faster than light, the event horizon must be a *null hypersurface*, a surface along which light travels.

- The definitions are really *global* statements, requiring knowledge of extended regions of spacetime. For, to know whether one is inside a black hole, one has to to know whether one can eventually escape, and that requires knowing one's future. Thus it is impossible to divine on the basis of local – here and now – measurements whether one is already inside a black hole. Indeed, the event horizon is an unmarked border, with no local signifiers of its presence such as a divergent curvature scalar. In fact, in nonstationary situations, an event horizon may be present even in flat space.

Now, since the inside of a black hole is causally disconnected from the outside, classical physics for an outside observer should be independent of the black hole's interior. One can therefore ask what an observer who always stays outside the horizon sees. As a number of authors have discovered, the answer is remarkable. The same horizon which we have just noted is invisible to an infalling observer, appears, to an outside observer, transformed into a dynamical membrane with tangible, *local* physical properties such as resistivity and viscosity. Moreover, the equations of motion governing the membrane are *nonrelativistic* – Ohm's law and the Navier-Stokes equation – even though they describe what is a quintessentially relativistic entity. Furthermore, these equations are dissipative, even though they may be derived in an action formulation. And the analogies with membranes go quite far; the horizon shares not only the classical properties of real membranes, but also their semi-classical ones. In particular, a membrane description of the horizon is able to account for such *thermodynamic* notions as entropy and the black hole's tendency to radiate as if it possessed a temperature. The main difference between horizon membranes and real membranes such as soap bubbles is that the black hole membrane is *acausal*, reacting to events before they occur. And, of course, it is not really there; as we have already noted, the observer who falls through the event horizon sees nothing, nothing at all.

That elaborate illusion is the main subject of this dissertation.

1.2 Background and Overview

The earliest relativistic studies of spherically symmetric gravitational sources began with the Schwarzschild line element,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{1.1}$$

from which it seemed that there was a singularity at the Schwarzschild radius, $r = 2M$. It was not realized initially that this singularity was merely a *coordinate singularity*, an artifact of using a pathological system of coordinates. Instead, it was thought that the region $r \leq 2M$ was somehow unphysical. Einstein himself contributed to this faulty view by showing that no stationary configuration of matter could exist inside the Schwarzschild radius, in the process repeating his cosmological "blunder" of overlooking nonstationary configurations (for a history, see [1]).

But later, with the work of Oppenheimer and Snyder, it became clear that a collapsing star could actually pass through the Schwarzschild radius and would effectively disappear from the outside, becoming a "black hole". Thus, Wheeler:

> ... it becomes dimmer millisecond by millisecond, and in less than a second is too dark to see. What was once the core of a star is no longer visible. The core like the Cheshire cat fades from view. One leaves

behind only its grin, the other, only its gravitational attraction ...

The idea that one could fall through the Schwarzschild radius opened up the study of the inside of black holes and of its relationship to the outside, a chain of work which culminated in the modern view of the event horizon as a causal boundary with the properties described above. Finally, in the work of Damour, Thorne, and others, the dynamical nature of the boundary, and its membrane interpretation, was clarified.

Our own study begins in Chapter 2 by considering these classical aspects of the event horizon. Actually, because null surfaces have a number of degenerate properties, mathematical descriptions of the event horizon are a little inconvenient. However, if, instead of the light-like event horizon, we consider a *time-like* surface infinitesimally outside it, then we can gain mathematical ease while still retaining the event horizon's other properties. This time-like surrogate, because it encloses the event horizon, is known as the *stretched horizon*, and it is this surface that we will be working with mostly. After collecting some mathematical facts about the stretched horizon, we demonstrate that its equations of motion can be derived from an action principle [2], and that they can be considered to describe a dynamical membrane. We provide concrete examples for the membrane interacting with electromagnetic, gravitational, and axidilaton fields, including derivations of Ohm's law, the Joule heating law, and the Navier-Stokes equation as they apply to black holes. This is followed by a discussion of the origin of dissipation, and its relation to the breaking of time-reversal symmetry. We conclude the chapter by providing a Hamiltonian formulation, and showing that, for quasi-stationary black holes, the equations of motion follow also from a minimum heat production principle, as advocated by

Prigogine.

When the fields the black hole interacts with are not classical but quantum fields, new phenomena emerge. Because the part of the wave function that lies within the black hole is inaccessible to the outside observer, we expect the black hole to have an entropy, associated with ignorance of those degrees of freedom. In addition, particles just inside the black hole can now escape, since the quantum uncertainty principle blurs the exact position of the horizon or, equivalently, of the particles. Thus black holes radiate [3]. Moreover, the spectrum of the outgoing radiation is, at least to first approximation, the Planckian spectrum of a thermal black-body. Hence one can associate a *temperature* to a black hole.

Historically, the concept of temperature and entropy for black holes was foreshadowed by several results in the classical theory [4, 5, 6, 7], in which something proportional to the surface gravity played the role of temperature, and some factor times the horizon's surface area mimicked the entropy. As we shall see the classical membrane equations similarly hint at thermodynamics. However, to go beyond analogies to actually identify the corresponding quantities requires quantum mechanics, and in particular a mechanism by which black holes can radiate.

Of course, escape from a black hole contradicts its very definition. It is easy to see that, when pushed to its logical limit, the whole concept of a black hole becomes unreliable, an essentially classical description. Thus, although quantum field theory around black holes does resolve some thermodynamic paradoxes, the tension between quantum mechanics and general relativity remains. In particular, a new *information puzzle* arises if the prediction of thermal radiation is taken literally. Specifically, the question arises whether the initial state that formed the black hole can be reconstructed from the outgoing radiation. The main problem here is this:

when matter falls into a black hole, the external configuration is determined by only a few quantities – this is the content of the "no-hair theorem" [8]. At this point, one could consider the information to reside inside the black hole. However, once the black hole starts to radiate, the inside disappears and one is essentially left with the outgoing radiation. But now purely thermal radiation is uncorrelated, so the outgoing radiation does not carry sufficient information to describe what made up the black hole.

If this is true, if the radiation is purely thermal as has been claimed [9], then physics loses its predictive power since the final state is not uniquely determined by the initial state, with the many different initial configurations of matter that could form a black hole all ending up as the same bath of thermal radiation. At stake in the information puzzle is the very idea that the past and the future are uniquely connected.

To decisively settle such questions, one really needs a quantum theory of gravity, one in which the quantum states that are counted by the black hole's entropy can be precisely enumerated, and any nonthermal aspects of black hole evaporation can be probed through scattering calculations. However, string theory, currently our only candidate theory of quantum gravity, is at present limited in its ability to answer some important questions about spacetime. Hence, in Chapter 3, we take an alternative approach: we truncate Einstein gravity at a semi-classical level, with the understanding that the infinite higher order corrections will actually be rendered finite by a suitable theory of quantum gravity. Within this approximation, we find continued success for the membrane action. By means of a Euclidean path integral, we find that the membrane action is also responsible for the black hole's Bekenstein-Hawking entropy, including the important numerical factor.

Next, we turn to black hole radiance. In previous derivations of Hawking radiation, the origin of the radiation has been somewhat obscure. Here we provide two short and physically appealing semi-classical derivations of Hawking radiation [10]. First, we show that the heuristic notion of black hole radiance as pair production through tunneling does indeed have quantitative support. The imaginary part of the action for tunneling across the horizon is related to the emission rate, just as in Schwinger pair creation in an electric field. Alternatively, an outside observer can consider the outgoing flux as consisting of spontaneous emissions from the membrane, rather than as particles that have tunneled across the horizon. Since we already have an action for the membrane, we can compute the rate for the membrane to shrink spontaneously; the emission rate agrees with the tunneling calculation.

We find that the probability for emission is approximately consistent with blackbody emission. However, unlike the original derivations of black hole radiance, our calculations respect energy conservation. Indeed, energy conservation is a fundamental requirement in tunneling, one that drives the dynamics, and without which our calculations do not even go through. The constraint that energy be conserved modifies the emission rate so that it is not *exactly* Planckian; there is a nonthermal correction to the spectrum. The correction to the Hawking formula is small when the outgoing particle carries away only a small fraction of the black hole's mass. However, it is qualitatively significant because nonthermality automatically implies the existence of correlations in the outgoing radiation, which means that at least some information must be returned.

Energy conservation has another, long-term, consequence: the black hole can actually disappear entirely if all its mass and charge are radiated away. The question of the causal structure of a spacetime containing an evaporating black hole is of some interest, not least because it is closely related to the possibility of information retrieval. In Chapter 4, we examine the causal structure of such a spacetime. Approximating the radiating black hole geometry by the Vaidya solution that describes the spacetime outside a radiating star, we construct a spacetime picture of the formation and subsequent evaporation of a charged black hole [11] in a special case. We find that the resultant Penrose diagram is predictable in the sense that post-evaporation conditions are causally dependent on initial conditions, a result consistent with information conservation.

Chapter 2

Classical Theory: The Membrane Paradigm

2.1 Introduction

The event horizon of a black hole is a peculiar object: it is a mathematically defined, locally undetectable boundary, a surface-of-no-return inside which light cones tip over and “time” becomes spatial (for a review see, e.g., [12]). Otherwise natural descriptions of physics often have trouble accommodating the horizon; as the most primitive example, the familiar Schwarzschild metric has a coordinate singularity there. Theories of fields that extend to the horizon face the additional challenge of having to define boundary conditions on a surface that is infinitely red-shifted, has a singular Jacobian, and possesses a normal vector which is also tangential. These considerations might induce one to believe that black hole horizons are fundamentally different from other physical entities.

On the other hand, further work has established a great variety of analogies between the horizon and more familiar, pre-relativistic bodies. In addition to the famous four laws of black hole thermodynamics [4, 5, 13, 6], which are global statements, there is also a precise local mechanical and electrodynamic correspondence. In effect, it has been shown [14, 15, 16, 17, 18] that an observer who remains outside a black hole perceives the horizon to behave according to equations that describe a fluid bubble with electrical conductivity as well as shear and bulk viscosities. Moreover, it is possible to define a set of local surface densities, such as charge or energy-momentum, which inhabit the bubble surface and which obey conservation laws. Quite remarkably, a general-relativistically *exact* calculation then leads, for arbitrary nonequilibrium black holes, to equations for the horizon which can be precisely identified with Ohm's law, the Joule heating law, and the Navier-Stokes equation.

These relations were originally derived for the mathematical, or true, event horizon. For astrophysical applications it became more convenient to consider instead a "stretched horizon," a (2+1)-dimensional time-like surface located slightly outside the true horizon. Because it has a nonsingular induced metric, the stretched horizon provides a more tractable boundary on which to anchor external fields; outside a complicated boundary layer, the equations governing the stretched horizon are to an excellent approximation [19, 20] the same as those for the true horizon. This view of a black hole as a dynamical time-like surface, or membrane, has been called the membrane paradigm [21].

Most of the mentioned results have been derived through general-relativistic calculations based on various intuitive physical arguments. In this chapter, we show that the gravitational and electromagnetic descriptions of the membrane can be derived systematically, directly, and more simply from the Einstein-Hilbert or Maxwell actions. Aside from the appeal inherent in a least action principle, an action formulation is a unifying framework which is easily generalizable and has the advantage of providing a bridge to thermodynamics and quantum mechanics (see [22] for related work).

The key idea in what follows is that, since (classically) nothing can emerge from a black hole, an observer who remains outside a black hole cannot be affected by the dynamics inside the hole. Hence the equations of motion ought to follow from varying an action restricted to the external universe. However, the boundary term in the derivation of the Euler-Lagrange equations does not in general vanish on the stretched horizon as it does at the boundary of spacetime. In order to obtain the correct equations of motion, we must add to the external action a surface term that cancels this residual boundary term. The membrane picture emerges in interpreting the added surface term as electromagnetic and gravitational sources residing on the stretched horizon.

2.2 Horizon Preliminaries

In this section, we fix our conventions, first in words, then in equations. Through every point on the true horizon there exists a unique null generator l^a which we may parameterize by some regular time coordinate whose normalization we fix to equal that of time-at-infinity. Next, we choose a time-like surface just outside the true horizon. This is the stretched horizon, $\mathcal{H}$, whose location we parameterize by $\alpha \ll 1$ so that $\alpha \to 0$ is the limit in which the stretched horizon coincides with the true horizon. We will always take this limit at the end of any computation. Since many of the useful intermediate quantities will diverge as inverse powers of α, we renormalize them by the appropriate power of α. In that sense, α plays the role of a regulator.

For our purposes, the principal reason for preferring the stretched horizon over the true horizon is that the metric on a time-like – rather than null – surface is nondegenerate, permitting one to write down a conventional action. Generically (in the absence of horizon caustics), a one-to-one correspondence between points on the true and stretched horizons is always possible via, for example, ingoing null rays that pierce both surfaces (see [20] for details).

We can take the stretched horizon to be the world-tube of a family of time-like observers who hover just outside the true horizon. These nearly light-like "fiducial" observers are pathological in that they suffer an enormous proper acceleration and measure quantities that diverge as $\alpha \to 0$. However, although we take the mathematical limit in which the true and stretched horizons conflate, for physical purposes the proper distance of the stretched horizon from the true horizon need only be smaller than the length scale involved in a given measurement. In that respect, the stretched horizon, although a surrogate for the true horizon, is actually more fundamental than the true horizon, since measurements at the stretched horizon constitute real measurements that an external observer could make and report, whereas accessing any quantity measured at the true horizon would entail the observer's inability to report back his or her results.

We take our fiducial observers to have world lines U^a, parameterized by their proper time, τ. The stretched horizon also possesses a space-like unit normal n^a which for consistency we shall always take to be outward-pointing. Moreover, we choose the normal vector congruence on the stretched horizon to emanate outwards along geodesics. We define α by requiring that $\alpha U^a \to l^a$ and $\alpha n^a \to l^a$; hence αU^a and αn^a are equal in the true horizon limit. This is nothing more than the statement that the null generator l^a is both normal and tangential to the true horizon, which is the defining property of null surfaces. Ultimately, though, it will be this property that will be responsible for the dissipative behavior of the horizons. The 3-metric, h_{ab}, on $\mathcal{H}$ can be written as a 4-dimensional tensor in terms of the spacetime metric and the normal vector, so that h^a_b projects from the spacetime tangent space to the 3-tangent space. Similarly, we can define the 2-metric, γ_{AB}, of the space-like section of $\mathcal{H}$ to which U^a is normal, in terms of the stretched horizon 3-metric and U^a, thus making a 2+1+1 split of spacetime. We denote the 4-covariant derivative by ∇_a, the 3-covariant derivative by $|a$, and the 2-covariant derivative by $\|A$. For a vector in the stretched horizon, the covariant derivatives are related by $h^c_d \nabla_c w^a = w^a_{|d} - K^c_d w_c n^a$ where $K^a_b \equiv h^c_b \nabla_c n^a$ is the stretched horizon's extrinsic curvature, or second fundamental form. In summary,

$$l^2 = 0 \tag{2.1}$$

$$U^a = \left(\frac{d}{d\tau}\right)^a , \quad U^2 = -1 , \quad \lim_{\alpha\to\infty} \alpha U^a = l^a \tag{2.2}$$

$$n^2 = +1 , \quad a^c = n^a \nabla_a n^c = 0 , \quad \lim_{\alpha\to\infty} \alpha n^a = l^a \tag{2.3}$$

$$h^a_b = g^a_b - n^a n_b , \quad \gamma^a_b = h^a_b + U^a U_b = g^a_b - n^a n_b + U^a U_b \tag{2.4}$$

$$K^a_b \equiv h^c_b \nabla_c n^a , \quad K_{ab} = K_{ba} , \quad K_{ab} n^b = 0 \tag{2.5}$$

$$w^c \epsilon \mathcal{H} \Rightarrow h^c_d \nabla_c w^a = w^a_{|d} - K^c_d w_c n^a \Rightarrow \nabla_c w^c = w^c_{|c} + w^c a_c = w^c_{|c} . \tag{2.6}$$

The last expression relates the covariant divergence associated with g_{ab} to the covariant divergence associated with h_{ab}.

For example, the Reissner-Nordström solution has

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 , \tag{2.7}$$

so that a stretched horizon at constant r would have

$$\alpha = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2} , \tag{2.8}$$

$$U_a = -\alpha \left(dt\right)_a \ , \tag{2.9}$$

and

$$n_a = +\alpha^{-1} \left(dr\right)_a \ . \tag{2.10}$$

2.3 Action Formulation

To find the complete equations of motion by extremizing an action, it is not sufficient to set the bulk variation of the action to zero: one also needs to use the boundary conditions. Here we take our Dirichlet boundary conditions to be $\delta\varphi = 0$ at the boundary of spacetime, where φ stands for any field.

Now since the fields inside a black hole cannot have any classical relevance for an external observer, the physics must follow from varying the part of the action restricted to the spacetime outside the black hole. However, this external action is not stationary on its own, because boundary conditions are fixed only at the singularity and at infinity, but not at the stretched horizon. Consequently, we rewrite the total action as

$$S_{\text{world}} = \left(S_{\text{out}} + S_{\text{surf}}\right) + \left(S_{\text{in}} - S_{\text{surf}}\right) \ , \tag{2.11}$$

where now $\delta S_{\text{out}} + \delta S_{\text{surf}} \equiv 0$, which implies also that $\delta S_{\text{in}} - \delta S_{\text{surf}} = 0$. The total action has been broken down into two parts, both of which are stationary on their own, and which do not require any new boundary conditions.

The surface term, $S_{\rm surf}$, corresponds to sources, such as surface electric charges and currents for the Maxwell action, or surface stress tensors for the Einstein-Hilbert action. The sources are fictitious: an observer who falls through the stretched horizon will not find any surface sources and, in fact, will not find any stretched horizon. Furthermore, the field configurations inside the black hole will be measured by this observer to be entirely different from those posited by the membrane paradigm. On the other hand, for an external fiducial observer the source terms are a very useful artifice; their presence is consistent with all external fields. This situation is directly analogous to the method of image charges in electrostatics, in which a fictitious charge distribution is added to the system to implement, say, conducting boundary conditions. By virtue of the uniqueness of solutions to Poisson's equation with conducting boundary conditions, the electric potential on one – and only one – side of the boundary is guaranteed to be the correct potential. An observer who remains on that side of the boundary has no way of telling through the fields alone whether they arise through the fictitious image charges or through actual surface charges. The illusion is exposed only to the observer who crosses the boundary to find that not only are there no charges, but the potential on the other side of the boundary is quite different from what it would have been had the image charges been real.

In the rest of this section, we shall implement Eq. (2.11) concretely in important special cases.

2.3.1 The Electromagnetic Membrane

The external Maxwell action is

$$S_{\mathrm{out}}[A_a] = \int d^4x\sqrt{-g}\,\left(-\frac{1}{16\pi}F^2 + J\cdot A\right)\ , \tag{2.12}$$

where F is the electromagnetic field strength. Under variation, we obtain the inhomogeneous Maxwell equations

$$\nabla_b F^{ab} = 4\pi J^a\ , \tag{2.13}$$

as well as the boundary term

$$\frac{1}{4\pi}\int d^3x\sqrt{-h}\,F^{ab}n_a\delta A_b\ , \tag{2.14}$$

where h is the determinant of the induced metric, and n^a is the outward-pointing space-like unit normal to the stretched horizon. We need to cancel this term. Adding the surface term

$$S_{\mathrm{surf}}[A_a] = +\int d^3x\sqrt{-h}\,j_s\cdot A\ , \tag{2.15}$$

we see that we must have

$$j_s^a = +\frac{1}{4\pi}F^{ab}n_b\ . \tag{2.16}$$

The surface 4-current, j_s^a, has a simple physical interpretation. We see that its time-component is a surface charge, σ, that terminates the normal component of the electric field just outside the membrane, while the spatial components, $\vec{j}_s$, form a surface current that terminates the tangential component of the external magnetic field:

$$E_\perp = -U_aF^{ab}n_b = 4\pi\sigma \tag{2.17}$$

$$\vec{B}_{\|}^A = \epsilon^A_B\gamma^B_a F^{ab}n_b = 4\pi\left(\vec{j}_s\times\hat{n}\right)^A\ . \tag{2.18}$$

It is characteristic of the membrane paradigm that σ and $\vec{j}_s$ are *local* densities, so that the total charge on the black hole is the surface integral of σ over the membrane, taken at some constant universal time. This is in contrast to the total charge of a Reissner-Nordström black hole, which is a global characteristic that can be defined by an integral at spatial infinity.

From Maxwell's equations and Eq. (2.16), we obtain a continuity equation for the membrane 4-current which, for a stationary hole, takes the form

$$\frac{\partial \sigma}{\partial \tau} + \vec{\nabla}_2 \cdot \vec{j}_s = -J^n \ , \tag{2.19}$$

where $\vec{\nabla}_2 \cdot \vec{j}_s \equiv \left(\gamma^A_a j^a_s\right)_{||A}$ is the two-dimensional divergence of the membrane surface current, and $-J^n = -J^a n_a$ is the amount of charge that falls into the hole per unit area per unit proper time, τ. Physically, this equation expresses local charge conservation in that any charge that falls into the black hole can be regarded as remaining on the membrane: the membrane is impermeable to charge.

The equations we have so far are sufficient to determine the fields outside the horizon, given initial conditions outside the horizon. A plausible requirement for initial conditions *at* the horizon is that the fields measured by freely falling observers (FFO's) at the stretched horizon be finite. There being no curvature singularity at the horizon, inertial observers who fall through the horizon should detect nothing out of the ordinary. In contrast, the fiducial observers (FIDO's) who make measurements at the membrane are infinitely accelerated. Their measurements, subject to infinite Lorentz boosts, are singular. For the electromagnetic fields we have, with γ the Lorentz boost and using orthonormal coordinates,

$$E_\theta^{\rm FIDO} \approx \gamma \left(E_\theta^{\rm FFO} - B_\varphi^{\rm FFO}\right) \ , \qquad B_\varphi^{\rm FIDO} \approx \gamma \left(B_\varphi^{\rm FFO} - E_\theta^{\rm FFO}\right) \ , \tag{2.20}$$

$$B_\theta^{\rm FIDO} \approx \gamma \left(B_\theta^{\rm FFO} - E_\varphi^{\rm FFO}\right) \ , \qquad E_\varphi^{\rm FIDO} \approx \gamma \left(E_\varphi^{\rm FFO} - B_\theta^{\rm FFO}\right) \ , \tag{2.21}$$

or, more compactly,

$$\vec{E}_{||}^{\rm FIDO} = \hat{n} \times \vec{B}_{||}^{\rm FIDO} \ . \tag{2.22}$$

That is, the regularity condition states that all radiation in the normal direction is ingoing; a black hole acts as a perfect absorber. Combining the regularity condition with Eq. (2.18) and dropping the FIDO label, we arrive at

$$\vec{E}_{||} = 4\pi \vec{j}_s \ . \tag{2.23}$$

That is, black holes obey Ohm's law with a surface resistivity of $\rho = 4\pi \approx 377\ \Omega$. Furthermore, the Poynting flux is

$$\vec{S} = \frac{1}{4\pi}\left(\vec{E} \times \vec{B}\right) = -j_s^2 \rho\, \hat{n} \ . \tag{2.24}$$

We can integrate this over the black hole horizon at some fixed time. However, for a generic stretched horizon, we cannot time-slice using fiducial time as different fiducial observers have clocks that do not necessarily remain synchronized. Consequently we must use some other time for slicing purposes, such as the time at infinity, and then include in the integrand a (potentially position-dependent) factor to convert the locally measured energy flux to one at infinity. With a clever choice of the stretched horizon, however, it is possible to arrange that all fiducial observers have synchronized clocks. In this case, two powers of α, which is now the lapse, are included in the integrand. Then, for some given universal time, t, the power radiated into the black hole, which is also the rate of increase of the black hole's irreducible mass, is given by

$$\frac{dM_{\rm irr}}{dt} = -\int \alpha^2 \vec{S} \cdot d\vec{A} = +\int \alpha^2 j_s^2 \rho\, dA \ . \tag{2.25}$$

That is, black holes obey the Joule heating law, the same law that also describes the dissipation of an ordinary Ohmic resistor.

2.3.2 The Gravitational Membrane

We turn now to gravity. The external Einstein-Hilbert action is

$$S_{\text{out}}[g^{ab}] = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,R + \frac{1}{8\pi}\oint d^3x\sqrt{\pm h}\,K + S_{\text{matter}}\ , \tag{2.26}$$

where R is the Ricci scalar, and K is the trace of the extrinsic curvature, and where for convenience we have chosen the field variable to be the inverse metric g^{ab}. The surface integral of K is only over the outer boundary of spacetime, and not over the stretched horizon. It is required in order to obtain the Einstein equations because the Ricci scalar contains second order derivatives of g_{ab}. When this action is varied, the bulk terms give the Einstein equations

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab}\ . \tag{2.27}$$

We are interested, however, in the interior boundary term. This comes from the variation of the Ricci tensor. We note that

$$g^{ab}\delta R_{ab} = \nabla^a\left[\nabla^b\left(\delta g_{ab}\right) - g^{cd}\nabla_a\left(\delta g_{cd}\right)\right]\ , \tag{2.28}$$

where $\delta g_{ab} = -g_{ac}g_{bd}\delta g^{cd}$. Gauss' theorem now gives

$$\int d^4x\sqrt{-g}\,\left(g^{ab}\delta R_{ab}\right) = -\int d^3x\sqrt{-h}\,n^a h^{bc}\left[\nabla_c\left(\delta g_{ab}\right) - \nabla_a\left(\delta g_{bc}\right)\right]\ , \tag{2.29}$$

where the minus sign arises from choosing n^a to be outward-pointing. Applying the Leibniz rule, we can rewrite this as

$$\begin{aligned}\int d^4x\sqrt{-g}\,\left(g^{ab}\delta R_{ab}\right) \ &= \ \int d^3x\sqrt{-h}\,h^{bc}\left[\nabla_a\left(n^a\delta g_{bc}\right) - \delta g_{bc}\nabla_a\left(n^a\right)\right.\\ &\qquad \left. - \nabla_c\left(n^a\delta g_{ab}\right) + \delta g_{ab}\nabla_c\left(n^a\right)\right]\ .\end{aligned} \tag{2.30}$$

Now, in the limit that the stretched horizon approaches the null horizon, the first and third terms on the right-hand side vanish:

$$\int d^3x\sqrt{-h}\,h^{bc}\left[\nabla_a\left(n^a\delta g_{bc}\right) - \nabla_c\left(n^a\delta g_{ab}\right)\right] = 0\ . \tag{2.31}$$

A proof of this identity is given in the Appendix. With $K^{ba} = h^{bc}\nabla_c n^a$, the variation of the external action is

$$\delta S_{\rm out}[g^{ab}] = \frac{1}{16\pi}\int d^3x\sqrt{-h}\,\left(Kh_{ab} - K_{ab}\right)\delta g^{ab}\,. \tag{2.32}$$

Since the expression in parentheses contains only stretched horizon tensors, the normal vectors in the variation $\delta g^{ab} = \delta h^{ab} + \delta n^a n^b + n^a \delta n^b$ contribute nothing. As in the electromagnetic case, we add a surface source term to the action to cancel this residual boundary term. The variation of the required term can therefore be written as

$$\delta S_{\rm surf}[h^{ab}] = -\frac{1}{2}\int d^3x\sqrt{-h}\,t_{s\,ab}\delta h^{ab}\,. \tag{2.33}$$

We shall see later that this variation is integrable; i.e., an action with this variation exists. Comparison with Eq. (2.32) yields the membrane stress tensor

$$t_s^{ab} = +\frac{1}{8\pi}\left(Kh^{ab} - K^{ab}\right)\,. \tag{2.34}$$

Now just as a surface charge produces a discontinuity in the normal component of the electric field, a surface stress term creates a discontinuity in the extrinsic curvature. The relation between the discontinuity and the source term is given by the Israel junction condition [23, 24],

$$t_s^{ab} = \frac{1}{8\pi}\left([K]h^{ab} - [K]^{ab}\right)\,, \tag{2.35}$$

where $[K] = K_+ - K_-$ is the difference in the extrinsic curvature of the stretched horizon between its embedding in the external universe and its embedding in the spacetime internal to the black hole. Comparing this with our result for the membrane stress tensor, Eq. (2.34), we see that

$$K_-^{ab} = 0\,, \tag{2.36}$$

so that the interior of the stretched horizon molds itself into flat space. The Einstein equations, Eq. (2.27), can be rewritten via the contracted Gauss-Codazzi equations [24] as

$$t_s^{ab}{}_{|b} = -h^a_c T^{cd} n_d \ . \tag{2.37}$$

Equations (2.34) and (2.37) taken together imply that the stretched horizon can be thought of as a fluid membrane, obeying the Navier-Stokes equation. To see this, recall that as we send α to zero, both αU^a and αn^a approach l^a, the null generator at the corresponding point on the true horizon. Hence, in this limit we can equate αU^a and αn^a, permitting us to write the relevant components of K^a_b, in terms of the surface gravity, g, and the extrinsic curvature, k^A_B, of a space-like 2-section of the stretched horizon:

$$U^c \nabla_c n^a \to \alpha^{-2} l^c \nabla_c l^a \equiv \alpha^{-2} g_H l^a \Rightarrow K^U_U = -g \ , \ \ K^A_U = \gamma^A_a K^a_b U^b = 0 \ , \tag{2.38}$$

where $g_H \equiv \alpha g$ is the renormalized surface gravity at the horizon, and

$$\gamma^c_A \nabla_c n^b \to \alpha^{-1} \gamma^c_A \nabla_c l^b \Rightarrow K^B_A = \gamma^a_A K^b_a \gamma^B_b = \alpha^{-1} k^B_A \ , \tag{2.39}$$

where k_{AB} is the extrinsic curvature of a space-like 2-section of the true horizon,

$$k_{AB} \equiv \gamma^d_A l_{B||d} = \frac{1}{2} \pounds_{l^a} \gamma_{AB} \ , \tag{2.40}$$

where $\pounds_{l^a}$ is the Lie derivative in the direction of l^a. We can decompose k_{AB} into a traceless part and a trace, $k_{AB} = \sigma_{AB} + \frac{1}{2}\gamma_{AB}\theta$, where σ_{AB} is the shear and θ the expansion of the world lines of nearby horizon surface elements. Then

$$t_s^{AB} = \frac{1}{8\pi} \left[-\sigma^{AB} + \gamma^{AB} \left(\frac{1}{2}\theta + g \right) \right] \ . \tag{2.41}$$

But this is just the equation for the stress of a two-dimensional viscous Newtonian fluid [25] with pressure $p = g/8\pi$, shear viscosity $\eta = 1/16\pi$, and bulk viscosity

$\zeta = -1/16\pi$. Hence we may identify the horizon with a two-dimensional dynamical fluid, or membrane. Note that, unlike ordinary fluids, the membrane has negative bulk viscosity. This would ordinarily indicate an instability against generic perturbations triggering expansion or contraction. It can be regarded as reflecting a null hypersurface's natural tendency to expand or contract [18]. Below we shall show how for the horizon this particular instability is replaced with a different kind of instability.

Inserting the A-momentum density $t_{s\,a}^{\;b}\gamma^a_A U_b = t_{s\;A}^{\;U} \equiv \pi_A$ into the Einstein equations, Eq. (2.37), we arrive at the Navier-Stokes equation

$$\pounds_\tau \pi_A = -\nabla_A p + \zeta \nabla_A \theta + 2\eta \sigma_{A\;\;||B}^{\;B} - T^n_A \;, \tag{2.42}$$

where $\pounds_\tau \pi_A = \partial \pi_A / \partial \tau$ is the Lie derivative (which is the general-relativistic equivalent of the convective derivative) with respect to proper time, and $-T^n_A = -\gamma^a_A T^c_a n_c$ is the flux of A-momentum into the black hole.

Inserting the U-momentum (energy) density $t_{s\;b}^{\;a} U_a U^b \equiv \Sigma = -\theta/8\pi$ gives

$$\pounds_\tau \Sigma + \theta\Sigma = -p\theta + \zeta\theta^2 + 2\eta\sigma_{AB}\sigma^{AB} + T^a_b n_a U^b \;, \tag{2.43}$$

which is the focusing equation for a null geodesic congruence [26]. We might now suspect that if the analogy with fluids extends to thermodynamics, then Eq. (2.43), as the equation of energy conservation, must be the heat transfer equation [25] for a two-dimensional fluid. Writing the expansion of the fluid in terms of the area, ΔA, of a patch,

$$\theta = \frac{1}{\Delta A}\frac{d\Delta A}{d\tau} \;, \tag{2.44}$$

we see that we can indeed rewrite Eq. (2.43) as the heat transfer equation (albeit with an additional relativistic term on the left)

$$T\left(\frac{d\Delta S}{d\tau}-\frac{1}{g}\frac{d^2\Delta S}{d\tau^2}\right)=\left(\zeta\theta^2+2\eta\sigma_{AB}\sigma^{AB}+T^a_b n_a U^b\right)\Delta A\,, \tag{2.45}$$

with T the temperature and S the entropy, provided that the entropy is given by

$$S=\eta\frac{k_B}{\hbar}A\,, \tag{2.46}$$

and the temperature by

$$T=\frac{\hbar}{8\pi k_B\eta}g\,, \tag{2.47}$$

where η is some proportionality constant.

Thus, the identification of the horizon with a fluid membrane can be extended to the thermodynamic domain. Nonetheless, the membrane is an unusual fluid. The focusing equation itself, Eq. (2.43), is identical in form to the equation of energy conservation for a fluid. However, because the energy density, Σ, is proportional to the expansion, $\Sigma=-\theta/8\pi$, one obtains a nonlinear first-order differential equation for θ which has no counterpart for ordinary fluids. The crucial point is that, owing to the black hole's gravitational self-attraction, the energy density is negative, and the solution to the differential equation represents a horizon that grows with time. For example, the source-free solution with a time-slicing for which the horizon has constant surface gravity is

$$\theta\left(t\right)=\frac{2g}{1+\left(\frac{2g}{\theta(t_0)}-1\right)e^{g(t_0-t)}}\,. \tag{2.48}$$

Because of the sign of the exponent, this would represent an ever-expanding horizon if $\theta\left(t_0\right)$ were an initial condition; the area of the horizon, which is related to θ by $\theta=(d/d\tau)\ln\sqrt{\gamma}$, expands exponentially with time. To avoid this runaway, one must impose "teleological boundary conditions" (that is, final conditions) rather than initial conditions. Hence, the horizon's growth is actually acausal; the membrane expands to intercept infalling matter that is yet to fall in [21]. This is because the membrane inherits the global character of the true horizon: the stretched horizon covers the true horizon whose location can only be determined by tracking null rays into the infinite future. In fact, the left-hand side of the heat transfer equation, Eq. (2.45), is of the same form as that of an electron subject to radiation reaction; the acausality of the horizon is therefore analogous to the pre-acceleration of the electron.

At this classical level, using only the equations of motion, the parameter η in Eq. (2.46) is undetermined. However, because we have an action we hope to do better, since the normalization in the path integral is now fixed. By means of a Euclidean path integral, we should actually be able to derive the Bekenstein-Hawking entropy, including the coefficient η, from the membrane action. We do this in the next chapter.

2.3.3 The Axidilaton Membrane

Another advantage of the action formulation is that it is easily generalized to arbitrary fields. For example, we can extend the membrane paradigm to include the basic fields of quantum gravity. Here we use the tree-level effective action obtained from string theory after compactification to four macroscopic dimensions. This action is a generalization of the classical Einstein-Hilbert-Maxwell action to which it reduces when the axidilaton, λ, is set to $i/16\pi$. The action is

$$S[\lambda, \overline{\lambda}, A_a, g_{ab}] = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi} - \frac{|\partial\lambda|^2}{2\lambda_2^2} + \frac{i}{4} \left(\lambda F_+^2 - \overline{\lambda} F_-^2 \right) \right) , \qquad (2.49)$$

where R is the four-dimensional Ricci curvature scalar, $F_\pm \equiv F \pm i\tilde{F}$ are the self- and anti-self-dual electromagnetic field strengths, and $\lambda \equiv \lambda_1 + i\lambda_2 = a + ie^{-2\varphi}$ is the axidilaton, with a the axion and φ the dilaton. Solutions to the equations of motion arising from this action include electrically (Reissner-Nordström) and magnetically charged black holes [27, 28], as well as their duality-rotated cousins, dyonic black holes [29],which carry both electric and magnetic charge.

The equations of motion are

$$\nabla_a \left(\frac{\partial^a \lambda}{\lambda_2^2} \right) + i \frac{|\partial\lambda|^2}{\lambda_2^3} - \frac{i}{2} F_-^2 = 0 \tag{2.50}$$

and

$$\nabla_a \left(\lambda F_+^{ab} - \overline{\lambda} F_-^{ab} \right) = 0 \ , \tag{2.51}$$

besides the Einstein equations.

As before, we require the external action to vanish on its own. Integration by parts on the axidilaton kinetic term leads to a variation at the boundary,

$$\int d^3x \sqrt{-h} \left[\delta\lambda \left(\frac{n_a \partial^a \overline{\lambda}}{2\lambda_2^2} \right) + \delta\overline{\lambda} \left(\frac{n_a \partial^a \lambda}{2\lambda_2^2} \right) \right] \ , \tag{2.52}$$

where n^a is again chosen to be outward-pointing. To cancel this, we add the surface term

$$S_{\rm surf} = \int d^3x \sqrt{-h} \, \left(\lambda \overline{q} + \overline{\lambda} q \right) \ , \tag{2.53}$$

so that

$$q = - \frac{n_a \partial^a \lambda}{\lambda_2^2} \ . \tag{2.54}$$

To interpret this, we note that the kinetic term in λ is invariant under global $SL(2, \mathbb{R})$ transformations of the form

$$\lambda \to \frac{a\lambda + b}{c\lambda + d} \ , \ \ ad - bc = 1 \ , \tag{2.55}$$

which are generated by Peccei-Quinn shifts, $\lambda_1 \to \lambda_1 + b$, and duality transformations, $\lambda \to -1/\lambda$. The Peccei-Quinn shift of the axion can be promoted to a classical local symmetry to yield a Nöther current:

$$J^a_{P-Q} = -\frac{1}{2\lambda_2^2}\left(\partial^a\lambda + \partial^a\overline{\lambda}\right) . \tag{2.56}$$

Therefore, under a Peccei-Quinn shift,

$$\delta S_{\text{surf}} = \int d^3x\sqrt{-h}\,\delta\lambda\,(q+\overline{q}) = \int d^3x\sqrt{-h}\,\delta\lambda\left(n_a J^a_{P-Q}\right) . \tag{2.57}$$

The sum of the q and $\overline{q}$ terms induced at the membrane, Eq. (2.54), is the normal component of the Peccei-Quinn current. Hence, at the membrane,

$$\left(h^a_b J^b_{P-Q}\right)_{|a} = -F\tilde{F} - \nabla_a\left[(q+\overline{q})\,n^a\right] . \tag{2.58}$$

That is, the membrane term $\nabla_a\left[(q+\overline{q})\,n^a\right]$ augments the dyonic $F\tilde{F}$ term as a source for the three-dimensional Peccei-Quinn current, $h^a_b J^b_{P-Q}$, at the membrane.

The membrane is again dissipative with the Peccei-Quinn charge accounting for the dissipation in the usual $\alpha \to 0$ limit. The local rate of dissipation is given by the bulk stress tensor at the membrane:

$$T_{ab}U^a n^b = \frac{1}{16\pi}\frac{\partial_a\lambda\partial_b\overline{\lambda} + \partial_a\overline{\lambda}\partial_b\lambda}{2\lambda_2^2}U^a n^b \to \frac{\lambda_2^2|q|^2}{16\pi} . \tag{2.59}$$

In addition, the presence of the axidilaton affects the electromagnetic membrane. (The gravitational membrane is unaffected since the surface terms come from the Ricci scalar which has no axidilaton factor.) The electromagnetic current is now

$$j^a_s = -2i\left(\lambda F^{ab}_+ - \overline{\lambda}F^{ab}_-\right)n_b . \tag{2.60}$$

The surface charge is therefore

$$\sigma = 4\left(\lambda_2 E_\perp + \lambda_1 B_\perp\right) , \tag{2.61}$$

and the surface current is

$$\vec{j}_s = 4\left(\lambda_2 \hat{n} \times \vec{B}_{||} - \lambda_1 \hat{n} \times \vec{E}_{||}\right) \ , \tag{2.62}$$

which, by the regularity of the electromagnetic field, Eq. (2.22), satisfies

$$\begin{pmatrix} j_s^\theta \\ j_s^\varphi \end{pmatrix} = 4 \begin{pmatrix} \lambda_2 & \lambda_1 \\ -\lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} E^\theta \\ E^\varphi \end{pmatrix} \ . \tag{2.63}$$

The conductivity is now a tensor. When the axion is absent, the resistivity is

$$\rho = \frac{1}{4\lambda_2} \ . \tag{2.64}$$

The inverse dependence on λ_2 is to be expected on dimensional grounds. The pure dilaton action can be derived from Kaluza-Klein compactification of pure gravity in five dimensions, where the fifth dimension is curled into a circle of radius $e^{-2\varphi} = \lambda_2$. In five dimensions, with $c \equiv 1$, resistance (and hence resistivity for a two-dimensional resistor such as the membrane) has dimensions of inverse length. Using the regularity condition, Eq. (2.22), the rate of dissipation, for a stretched horizon defined to have uniform lapse α with respect to time at infinity, t, is

$$\frac{dM_{\rm irr}}{dt} = -\int \alpha^2 \vec{S} \cdot d\vec{A} = \int 4\alpha^2 \lambda_2 E_{||}^2 dA = \int \alpha^2 \frac{\lambda_2}{4|\lambda|^4} \vec{j}_s^{\,2} dA \ , \tag{2.65}$$

which is the Joule heating law in the presence of an axidilaton.

2.4 Dissipation

Given that the bulk equations of motion are manifestly symmetric under time-reversal, the appearance of dissipation, as in Joule heating and fluid viscosity, might seem mysterious, all the more so since it has been derived from an action.

The procedure, described here, of restricting the action to some region and adding surface terms on the boundary of the region cannot be applied with impunity to any arbitrary region: a black hole is special. This is because the region outside the black hole contains its own causal past; an observer who remains outside the black hole is justified in neglecting (indeed, is unaware of) events inside. However, even "past sufficiency" does not adequately capture the requirements for our membrane approach. For instance, the past light cone of a spacetime point obviously contains its own past, but an observer in this light cone must eventually leave it. Rather, we define the notion of a future dynamically closed set:

> A set S in a time-orientable globally hyperbolic spacetime (M, g_{ab}) is *future dynamically closed* if $J^{-}(S) = S$, and if, for some foliation of Cauchy surfaces Σ_t parameterized by the values of some global time function, we have that $\forall\, t_0\, \forall\, p \,\epsilon\, (S \cap \Sigma_{t_0})\ \forall\, (t > t_0)\ \exists\, q \,\epsilon\, (I^{+}(p) \cap S \cap \Sigma_t)$.

That is, S is future dynamically closed if it contains its own causal past and if from every point in S it is possible for an observer to remain in S. Classically, the region outside the true horizon of a black hole is dynamically closed. So too is the region on one side of a null plane in flat space; this is just the infinite-mass limit of a black hole. The region outside the stretched horizon is strictly speaking *not* dynamically closed since a signal originating in the thin region between the stretched horizon and the true horizon can propagate out beyond the stretched horizon. However, in the limit that the stretched horizon goes to the true horizon, $\alpha \to 0$, this region becomes vanishingly thin so that in this limit, which is in any case assumed throughout, we are justified in restricting the action.

The breaking of time-reversal symmetry comes from the definition of the stretched horizon; the region exterior to the black hole does not remain future dynamically closed under time-reversal. In other words, we have divided spacetime into two regions whose dynamics are derived from two different simultaneously vanishing actions, $\delta\left(S_{\mathrm{out}}+S_{\mathrm{surf}}\right)=\delta\left(S_{\mathrm{in}}-S_{\mathrm{surf}}\right)=0$. Given data on some suitable achronal subset we can, for the exterior region, predict the future but not the entire past, while, inside the black hole, we can "postdict" the past but cannot determine the entire future. Thus, our choice of the horizon as a boundary implicitly contains the irreducible logical requirement for dissipation, that is, asymmetry between past and future.

Besides the global properties that logically permit one to write down a time-reversal asymmetric action, there is also a local property of the horizon which is the proximate cause for dissipation, namely that the normal to the horizon is also tangential to the horizon. Without this crucial property – which manifests itself as the regularity condition, or the identification of the stretched horizon extrinsic curvature with intrinsic properties of the true horizon – there would still be surface terms induced at the stretched horizon, but no dissipation.

The regularity condition imposed at the boundary is not an operator identity, but a statement about physical states: all tangential electromagnetic fields as measured by a fiducial observer must be ingoing. Such a statement is not rigorously true. For any given value of $\alpha=(1-2M/r)^{1/2}$, there is a maximum wavelength, $\lambda_{\max}$, for outgoing modes that are invisible to the observer:

$$\lambda_{\max}=\frac{r-2M}{(1-2M/r)^{1/2}}\rightarrow 2M\alpha\ . \tag{2.66}$$

Dissipation occurs in the membrane paradigm because the finite but very high-frequency modes that are invisible to the fiducial observer are tacitly assumed not to exist. The regularity condition amounts to a coarse-graining over these modes.

We conclude this section with an illustration of the intuitive advantage of the membrane paradigm. It is a famous result that the external state of a stationary black hole, quite unlike that of other macroscopic bodies, can be completely characterized by only four quantities: the mass, the angular momentum, and the electric and magnetic charges. That such a "no-hair theorem" [8] should hold is certainly not immediately apparent from other black hole viewpoints. In the membrane picture, however, we can see this fairly easily. For example, an electric dipole that falls into the black hole can now be considered as merely two opposite charges incident on a conducting surface. The charges cause a current to flow and the current eventually dissipates; in the same way, all higher multipole moments are effaced. Similarly, the gravitational membrane obeys the Navier-Stokes equation, which is also dissipative; higher moments of an infalling mass distribution are thus obliterated in the same way. The only quantities that survive are those protected by the conservation laws of energy, angular momentum, and electric and magnetic charge. While far from an actual proof, this is at least a compelling and physically appealing argument for why black holes have only four "hairs."

2.5 Hamiltonian Formulation

The equations of motion can equally well be derived within a Hamiltonian formulation. This involves first singling out a global time coordinate, t, for the external universe, which is then sliced into space-like surfaces, Σ_t, of constant t. We can write, in the usual way,

$$t^a \equiv \left(\frac{d}{dt}\right)^a = \alpha U^a - v^a \ , \tag{2.67}$$

where U^a is the unit normal to Σ_t, $U^2 = -1$, and α and $-v^a$ are Wheeler's lapse and shift, respectively, with $v^a = dx^a/dt$ the ordinary 3-velocity of a particle with world-line U^a. For convenience we choose the stretched horizon to be a surface of constant lapse so that α, which goes to zero at the true horizon, serves as the stretched horizon regulator. The external Hamiltonian for electrodynamics, obtained from the Lagrangian via a Legendre transform and written in ordinary three-dimensional vector notation, is

$$H_{\rm out}[\varphi, \vec{A}, \vec{\pi}] = \frac{1}{4\pi}\int_{\Sigma_t} d^3x\sqrt{{}^3g}\left(\frac{1}{2}\alpha\left(\vec{E}\cdot\vec{E}+\vec{B}\cdot\vec{B}\right)+\vec{v}\cdot\left(\vec{E}\times\vec{B}\right)-\varphi\left(\vec{\nabla}\cdot\vec{E}\right)\right)\ , \tag{2.68}$$

where ${}^3g_{ab}$ is the 3-metric on Σ_t, $\varphi \equiv -A_a t^a$ is the scalar potential, $\vec{A}_a \equiv {}^3g_a^{\ b}A_b$ is the three-dimensional vector potential, and $\vec{\pi}^a \equiv -\sqrt{{}^3g}\vec{E}^a$ its canonical momentum conjugate. Note that $E^a = F^{ab}U_b$ is the co-moving electric field; $\vec{E}$ and $\vec{B}$ above refer to the fields measured by a fiducial observer with world-line U^a. Finally, the scalar potential is nondynamical; its presence in the Hamiltonian serves to enforce Gauss' law as a constraint. The equations of motion are now determined by Hamilton's equations and the constraint:

$$\frac{\delta H}{\delta\vec{\pi}} = \dot{\vec{A}}\ ,\quad \frac{\delta H}{\delta\vec{A}} = -\dot{\vec{\pi}}\ ,\quad \frac{\delta H}{\delta\varphi} = 0\ . \tag{2.69}$$

In the bulk these equations are simply Maxwell's equations but, because of the inner boundary, the usually discarded surface terms that arise during integration by parts now need to be canceled. It is easy to show then that the above equations hold only if additional surface terms are added to the Hamiltonian:

$$H = H_{\rm out} - \int d^2x\sqrt{\gamma}\, j_s\cdot A\ . \tag{2.70}$$

For Maxwell's equations to be satisfied in the bulk, the surface terms are once again the surface charges and currents necessary to terminate the normal electric and tangential magnetic fields at the stretched horizon. Thus, the membrane paradigm is recovered.

However, it is perhaps more interesting to proceed in a slightly different fashion. Instead of adding new terms, we can use the external Hamiltonian to prove the validity of a principle of minimum heat production. Such a principle, which holds under rather general circumstances for stationary dissipative systems, holds for black holes also in slightly nonstationary situations.

Now the time derivative of the external Hamiltonian is not zero, again because of the inner boundary. We can use Hamilton's equations to substitute expressions for the time derivative of the field and its momentum conjugate. Hamilton's equations are

$$\dot{\vec{A}} = -\alpha\vec{E} + \vec{v}\times\vec{B} - \vec{\nabla}\varphi \tag{2.71}$$

$$\dot{\vec{E}} = \vec{\nabla}\times\left(\alpha\vec{B} + \vec{v}\times\vec{E}\right) \ , \tag{2.72}$$

so that, making repeated use of the vector identity

$$\vec{\nabla}\cdot\left(\vec{V}\times\vec{W}\right) = \vec{W}\cdot\left(\vec{\nabla}\times\vec{V}\right) - \vec{V}\cdot\left(\vec{\nabla}\times\vec{W}\right) \ , \tag{2.73}$$

we find that the energy loss is

$$-\dot{H} = -\frac{1}{4\pi}\int d^2x\sqrt{\gamma}\,\left[\hat{n}\cdot\left(\alpha\vec{E}_{||}\times\alpha\vec{B}_{||}\right) + \vec{v}\cdot\left(E_{\perp}\alpha\vec{E}_{||} + B_{\perp}\alpha\vec{B}_{||}\right)\right] \ . \tag{2.74}$$

So far, we have used only Hamilton's equations. It remains, however, to implement the constraint. Hence we may regard $-\dot{H}$ as a functional of the Lagrange multiplier, φ. We therefore have

$$-\frac{\delta\dot{H}}{\delta\varphi} = -\frac{d}{dt}\frac{\delta H}{\delta\varphi} = 0 \ . \tag{2.75}$$

That is, the equations of motion follow from minimizing the rate of mass increase of the black hole with respect to the scalar potential. This is an exact statement;

we now show that this reduces to a minimum heat production principle in a quasi-stationary limit. Now we note that the first law of black hole thermodynamics allows us to decompose the mass change into irreducible and rotational parts:

$$\frac{dM}{dt} = \frac{dQ}{dt} + \Omega_H \frac{dJ}{dt} , \tag{2.76}$$

where Ω_H is the angular velocity at the horizon, and J is the hole's angular momentum. Since $|\vec{v}| \to \Omega_H$ at the horizon, we see that the second term on the right in Eq. (2.74) corresponds to the torquing of the black hole. When this is small, we may approximate the mass increase as coming from the first, irreducible term. Hence, in the quasi-stationary limit, for a slowly rotating black hole, the black hole's rate of mass increase is given by the dissipation of external energy. Invoking the regularity condition, Eq. (2.22), then gives

$$D[\varphi] = +\frac{1}{4\pi} \int d^2x \sqrt{\gamma} \left(\alpha \vec{E}_{||}\right)^2 , \quad \frac{\delta D}{\delta \varphi} = 0 , \tag{2.77}$$

where $\alpha \vec{E}_{||}$ is given by Eq. (2.71). This is the principle of minimum heat production: minimizing the dissipation functional leads to the membrane equation of motion.

We observe that we could have anticipated this answer. The numerical value of the Hamiltonian is the total energy of the system as measured at spatial infinity (assuming an asymptotically flat spacetime). The time derivative is then simply the rate, as measured by the universal time of distant observers, that energy changes. The rate of decrease of energy is the integral of the Poynting flux as measured by local observers, multiplied by two powers of α, one power to convert local energy to energy-at-infinity and one power to convert the rate measured by local clocks to the rate measured at infinity. Thus we can immediately define a dissipation functional:

$$D[\varphi] \equiv -\frac{1}{4\pi} \int d^2x \sqrt{\gamma}\, \hat{n} \cdot \left(\vec{E}_H \times \vec{B}_H\right) , \tag{2.78}$$

where the subscript H denotes that a power of α has been absorbed to renormalize an otherwise divergent fiducial quantity.

In this manner, we can easily write down the dissipation functional for gravity for which time-differentiating the Hamiltonian is a much more laborious exercise. The local rate of energy transfer is given by the right-hand side of the heat transfer equation, Eq. (2.45). The Hamiltonian for gravity satisfies two constraint equations with the lapse and shift vector serving as Lagrange multipliers. Since the membrane picture continues to have a gauge freedom associated with time-slicing, the constraint equation associated with the lapse is not implemented. This implies that the dissipation is a functional only of the shift. Hence we have

$$D[v^A] = \int d^2x\sqrt{\gamma}\left(\zeta\theta_H^2 + 2\eta\sigma_H^2 + \alpha^2 T^a_b n_a U^b\right) \ , \tag{2.79}$$

where again the two powers of α have been absorbed to render finite the quantities with the subscript H. Extremizing D with respect to v^A leads to the membrane equations of motion, enforcing the gauge constraint or, equivalently, obeying the principle of minimum heat production.

2.6 Appendix

In this appendix, we shall prove that Eq. (2.31) is zero in the limit that the stretched horizon approaches the true horizon. In that limit, $\alpha n^a \to l^a$. We shall make liberal use of Gauss' theorem, the Leibniz rule, and the fact that $h^{ab}n_b = K^{ab}n_b = 0$. In order to use Gauss' theorem, we note that since the "acceleration" $a^c \equiv n^d\nabla_d n^c$ of the normal vector (not to be confused with the fiducial acceleration $U^d\nabla_d U^c$) is zero, the 4-covariant divergence and the 3-covariant divergence of a vector in the stretched horizon are equal, Eq. (2.6).

Now, variations in the metric that are in fact merely gauge transformations can be set to zero. Using a vector v^a where v^a vanishes on the stretched horizon, we can gauge away the variations in the normal direction so that $\delta g_{ab} \to \delta h_{ab}$. Then the left-hand side of Eq. (2.31) becomes

$$
\begin{aligned}
&\int d^3x\sqrt{-h}\, h^{bc}\left[\nabla_a\left(n^a\delta h_{bc}\right) - \nabla_c\left(n^a\delta h_{ab}\right)\right] \\
&= \int d^3x\sqrt{-h}\left[\nabla_a\left(h^{bc}n^a\delta h_{bc}\right) - \left(\nabla_a h^{bc}\right)n^a\delta h_{bc} - \nabla_c\left(h^{bc}n^a\delta h_{ab}\right) + \left(\nabla_c h^{bc}\right)n^a\delta h_{ab}\right] \\
&= \int d^3x\sqrt{-h}\left[\nabla_a\left(h^{bc}n^a\delta h_{bc}\right) + \left(n^c a^b + n^b a^c\right)\delta h_{bc} - \left(h^{bc}n^a\delta h_{ab}\right)_{|c}\right. \\
&\qquad\left. -h^{bc}n^a\delta h_{ab}a_c - Kn^b n^a\delta h_{ab} - a^b n^a\delta h_{ab}\right] \\
&\qquad \text{(using } h^{bc} = g^{bc} - n^b n^c,\ K_{ab} = +h^c_a\nabla_c n_b, \text{ and } \nabla_c w^c = w^c_{|c} + w^c a_c \text{ for } w^c \epsilon \mathcal{H}) \\
&= \int d^3x\sqrt{-h}\left[\nabla_a\left(h^{bc}n^a\delta h_{bc}\right) - Kn^b n^a\delta h_{ab}\right] \\
&\qquad \text{(using Gauss' theorem, and } a^c = 0) \\
&= \int d^3x\sqrt{-h}\left[\nabla_a\left(h^{bc}\frac{\alpha}{\alpha}n^a\delta h_{bc}\right) - K\left[\delta\left(n^b n^a h_{ab}\right) - n^a h_{ab}\delta n^b - n^b h_{ab}\delta n^a\right]\right] \\
&\to \int d^3x\sqrt{-h}\,\nabla_a\left(h^{bc}\frac{1}{\alpha}l^a\delta h_{bc}\right) \text{(using } h_{ab}n^b = 0, \text{ and } \alpha n^a \to l^a) \\
&= \int d^3x\sqrt{-h}\left(h^{bc}\frac{1}{\alpha}l^a\delta h_{bc}\right)_{|a} \\
&= 0\,.
\end{aligned}
\tag{2.80}
$$

Chapter 3

Semi-Classical Theory: Thermodynamics

3.1 Introduction

A strong hint that the so far classical membrane analogy might remain valid at a thermodynamic level has already come from the focusing equation, Eq. (2.43), which we saw could be written as the heat transfer equation, Eq. (2.45), provided that the temperature was identified with the surface gravity and the entropy with the area. However, there are proportionality factors that cannot be determined through classical arguments alone. In this chapter we shall first show that the correct entropy with the correct numerical factor does indeed appear from the membrane action, though with a somewhat surprising sign.

One way to interpret the black hole's entropy is as the logarithm of the number of modes that propagate along the thin layer between the true horizon and the stretched horizon. The regularity condition, which led to dissipation even in the classical theory, essentially coarse-grained over these high-frequency modes. However, it is conceivable that in a quantum theory with benign ultraviolet behavior the amount of information contained in that region is finite. Einstein gravity is not such a theory but one may still ask abstractly whether an effective horizon theory could exist at a quantum level [30, 31]. Quantum effects cause the black hole to emit radiation. In order to preserve time-evolution unitarity, one might require the emitted radiation to be correlated with the interior state of the black hole. In this case, the membrane viewpoint remains valid only as a classical description, since quantum-mechanically the external universe receives information from the black hole in the form of deviations of the radiation from thermality; the crucial premise that the outside universe is emancipated from the internal state of the black hole is violated. It is important to emphasize, however, that correlations between the radiation and the horizon itself (as opposed to the inside of the hole) do not preclude the membrane paradigm. Indeed, the fact that the Bekenstein-Hawking entropy is proportional to the surface area of the black hole suggests that, even at the quantum level, an effective horizon theory may not be unfeasible.

3.2 String Theory or Field Theory?

In treating the quantum mechanical aspects of black holes, one is faced with a choice of two different approaches. One of the most exciting recent developments in theoretical physics has been the series of spectacular resolutions that string theory has brought to some longstanding problems in black hole physics. In particular, by constructing black hole solutions out of collections of D-branes, Strominger and Vafa were able to provide a microscopic account of black hole entropy in terms of excitations in higher, compactified dimensions [32]. On the heels of this triumph, came a description of black hole radiation as the emission of closed strings from the D-brane configuration [33]. Moreover, this process had an underlying unitary theory, indicating that information might not be lost.

These successes and the ultraviolet finiteness of string theory would indicate that a study of the quantum properties of black holes should proceed within the framework of string theory and indeed, in the long run, this may be true. However, at present, there are several limitations in the string theoretic approach to black holes that support continuing a field theoretic investigation. For example, the string calculations are only reliable for very special black holes (supersymmetric, four or five dimensional, extremal or near-extremal). Also the methods are not very general, with the entire calculation having to be repeated for each case, though the answer for the entropy – one-fourth the area – is simple enough. By contrast, the field theory calculation is short and valid at once for all black holes in any number of dimensions. Similarly, for black hole radiance, string theory has not progressed much beyond confirming field theory results, despite its unitary promise. In particular, statements about the changing nature of spacetime are difficult to make in string theory because the calculations are not controllable in the regime in which there is a classical spacetime. By contrast, in the next chapter, we shall obtain a Penrose diagram depicting the causal features of the spacetime. Finally, since the horizon can be in essentially flat space, it is not obvious that black hole radiance necessarily calls for a quantum theory of gravity (although see [34]). For these reasons, we shall take a field theory approach in this thesis.

Before moving on to the calculations, we mention as an aside that there are some intriguing parallels between the matrix formulation of M theory [35] and the membrane paradigm: both have a kind of holographic principle [36, 37], both have

Galilean equations emerging from a Lorentz-invariant theory, and in both the appearance of locality is somewhat mysterious. Quite possibly at least some of these similarities are related to the fact that both are formulated on null surfaces.

3.3 Entropy

We have mentioned that black hole entropy was something to be expected, since the part of the wavefunction or density matrix that lies within the black hole is inaccessible and must be traced over. Thus, the entropy can be thought of as originating in correlations across the horizon. This entanglement, or geometric, entropy may be computed in field theory [38, 39]. However, the field theory computations run aground because of uncontrollable ultraviolet divergences, and so this is not the approach taken here. Instead, we work with the path integral, making contact with thermodynamics by performing an analytic continuation to imaginary time, $\tau = it$, so that the path integral of the Euclideanized action becomes a partition function. For a stationary hole, regularity (or the removal of a conical singularity) dictates a period $\beta = \int d\tau = 2\pi/g_H$ in imaginary time [40], where g_H is the surface gravity; for a Schwarzschild hole, $\beta = 8\pi M$. This is the inverse Hawking temperature in units where $\hbar = c = G = k_B = 1$. The partition function is then the path integral over all Euclidean metrics which are periodic with period $2\pi/g_H$ in imaginary time. We can now evaluate the partition function in a stationary phase approximation:

$$Z = \int Dg_E^{ab} \exp\left(-\frac{1}{\hbar}\left(S^E_{\mathrm{out}}[g_E^{ab}] + S^E_{\mathrm{surf}}[h_E^{ab}]\right)\right) \approx \exp\left(-\frac{1}{\hbar}\left(S^E_{\mathrm{out}}[g^{ab}_{E\,\mathrm{cl}}] + S^E_{\mathrm{surf}}[h^{ab}_{E\,\mathrm{cl}}]\right)\right). \tag{3.1}$$

The external action itself can be written as $S_{\mathrm{out}} = S_{\mathrm{bulk}} + S_\infty$, where S_{bulk} is zero for a black hole alone in the universe. The boundary term S_∞ is the integral of the extrinsic curvature of the boundary of spacetime. In fact, a term proportional to the surface area at infinity can be included in S_∞ without affecting the Einstein equations since the metric is held fixed at infinity during variation. In particular, the proportionality constant can be chosen so that the action for all of spacetime is zero for Minkowski space:

$$S_\infty = \frac{1}{8\pi}\int d^3x\sqrt{-h}\,[K]\;, \tag{3.2}$$

where $[K]$ is the difference in the trace of the extrinsic curvature at the spacetime boundary for the metric g_{ab} and the flat-space metric η_{ab}. With this choice, the path integral has a properly normalized probabilistic interpretation. The Euclideanized value of S_∞ for the Schwarzschild solution is then [40]

$$S^E_\infty = \lim_{r\to\infty}\frac{1}{8\pi}\left(-32\pi^2 M\right)\left[(2r-3M) - 2r\left(1-\frac{2M}{r}\right)^{1/2}\right] = +4\pi M^2\;. \tag{3.3}$$

To obtain an explicit action for the membrane, we must integrate its variation, Eq. (2.33):

$$\delta S_{\text{surf}}[h^{ab}] = -\frac{1}{16\pi}\int d^3x\sqrt{-h}\;(K h_{ab} - K_{ab})\,\delta h^{ab}\;. \tag{3.4}$$

We see that

$$S_{\text{surf}}[h^{ab}] = \int d^3x\sqrt{-h}\;\left(B_{ab}h^{ab} - b\right) \tag{3.5}$$

is a solution, provided that the (undifferentiated) source terms are $B_{ab} = (+1/16\pi)K_{ab}$ and $b = (-1/16\pi)K$. This action has the form of surface matter plus a negative cosmological constant in three dimensions. The value of the membrane action for a solution to the classical field equations is then

$$S_{\text{surf}}[h^{ab}_{\text{cl}}] = +\frac{1}{8\pi}\int d^3x\sqrt{-h_{\text{cl}}}K_{\text{cl}}\;. \tag{3.6}$$

To evaluate this, we can take our fiducial world-lines U^a to be normal to the isometric time-slices of constant Schwarzschild time. The stretched horizon is then a surface of constant Schwarzschild r. Hence $\alpha = (1 - 2M/r)^{1/2}$, $\theta = 0$, and $K = g + \theta = g$, the unrenormalized surface gravity of the stretched horizon. Inserting these into Eq. (3.6), we find that the Euclidean action is

$$S^E_{\text{surf}} = \lim_{r \to r_H} \frac{1}{8\pi} \left(\int -d\tau \right) \alpha 4\pi r^2 g = -\pi {r_H}^2 = -4\pi M^2 \ , \tag{3.7}$$

where $r_H = 2M$ is the black hole's radius, and $g_H = \alpha g = 1/4M$ is its renormalized surface gravity.

The Euclidean membrane action exactly cancels the external action, Eq. (3.3). Hence the entropy is zero! That, however, is precisely what makes the membrane paradigm attractive: to an external observer, there is no black hole – only a membrane – and so neither a generalized entropy nor a strictly obeyed second law of thermodynamics. The entropy of the outside is simply the logarithm of the number of quantum states of the matter outside the membrane. This number decreases as matter leaves the external system to fall through and be dissipated by the membrane. When all matter has fallen into the membrane, the outside is in a single state – vacuum – and has zero entropy, as above.

To recover the Bekenstein-Hawking entropy, we must then use not the combination of external and membrane actions, which gave the entropy of the external system, but the combination of the *internal* and membrane actions,

$$Z_{B-H} = \int Dg_E^{ab} \exp \left(-\frac{1}{\hbar} \left(S^E_{\text{in}}[g_E^{ab}] - S^E_{\text{surf}}[h_E^{ab}] \right) \right) \ , \tag{3.8}$$

where now S_{surf} is subtracted [see Eq. (2.11)]. With $S_{\text{in}} = \int d^4x \sqrt{-g}\, R = 0$, the partition function for a Schwarzschild hole in the stationary phase approximation is

$$Z_{B-H} \approx \exp \left(-\frac{1}{\hbar} \left(+4\pi M^2 \right) \right) \ , \tag{3.9}$$

from which the Bekenstein-Hawking entropy, S_{B-H}, immediately follows:

$$S_{B-H} = \beta \left(M + \frac{\ln Z_{B-H}}{\beta} \right) = 8\pi M \left(M - \frac{1}{8\pi M} 4\pi M^2 \right) = \frac{1}{4} A \, , \tag{3.10}$$

which is the celebrated result.

For more general stationary (Kerr-Newman) holes, the Helmholtz free energy contains additional "chemical potential" terms corresponding to the other conserved quantities, Q and J,

$$F = M - TS - \Phi Q - \Omega J \, , \tag{3.11}$$

where $\Phi = Q/r_+$ and $\Omega = J/M$, where r_+ is the Boyer-Lindquist radial coordinate at the horizon. For a charged hole, the action also contains electromagnetic terms. The surface electromagnetic term, Eq. (2.15), has the value $(1/4\pi) \int d^3x \sqrt{-h}\, F^{ab} A_a n_b$. However, in order to have a regular vector potential, we must gauge transform it to $A_a = (Q/r - \Phi) \nabla_a t$ which vanishes on the surface. Hence, the surface action is again given by the gravitational term, which has the Euclideanized value $S^E_{\mathbf{surf}} = -\pi r_+^2$. It is easy to verify using Eq. (3.11) that this again leads to a black hole entropy equal to one-fourth of the horizon surface area and an external entropy of zero.

For nonstationary black holes, the extrinsic curvature also includes a term for the expansion of the horizon, $K = g + \theta$. Inserting this into the surface action enables us to calculate the instantaneous entropy as matter falls into the membrane in a nonequilibrium process. Of course, like the horizon itself, the entropy grows acausally.

3.4 Pictures of Hawking Radiation

We now turn to the phenomenon of black hole radiance. Although several derivations of Hawking radiation exist in the literature [3, 40], none of them correspond very directly to either of the two heuristic pictures that are most commonly proposed as ways to visualize the source of the radiation. According to one picture, the radiation arises by a process similar to Schwinger electron-positron pair creation in a constant electric field. The idea is that the energy of a particle changes sign as it crosses the horizon, so that a pair created just inside or just outside the horizon can materialize with zero total energy, after one member of the pair has tunneled to the opposite side. In the second picture, we work with the effective membrane representation of the horizon. Hawking radiation is then a special property of the membrane: its tendency towards spontaneous emission, as if it had a nonzero temperature.

Here we shall show that either of these pictures can in fact be used to provide short, direct semi-classical derivations of black hole radiation. In both cases energy conservation plays a fundamental role; one must make a transition between states with the same total energy, and the mass of the residual hole must go down as it radiates. Indeed, it is precisely the possibility of lowering the black hole mass which ultimately drives the dynamics. This supports the idea that, in quantum gravity, black holes are properly regarded as highly excited states.

In the standard calculation of Hawking radiation the background geometry is considered fixed, and energy conservation is not enforced. (The geometry is not truly static, despite appearances, as there is no global Killing vector.) Because we are treating this aspect more realistically, we must – and do – find corrections to the standard results. These become quantitatively significant when the quantum of radiation carries a substantial fraction of the mass of the hole.

3.5 Tunneling

To describe across-horizon phenomena, it is necessary to choose coordinates which, unlike Schwarzschild coordinates, are not singular at the horizon. A particularly suitable choice is obtained by introducing a time coordinate,

$$t = t_s + 2\sqrt{2Mr} + 2M \ln \frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}} , \tag{3.12}$$

where t_s is Schwarzschild time. With this choice, the line element reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + 2\sqrt{\frac{2M}{r}}\, dt\, dr + dr^2 + r^2 d\Omega^2 . \tag{3.13}$$

There is now no singularity at $r = 2M$, and the true character of the spacetime, as being stationary but not static, is manifest. These coordinates were first introduced by the French mathematician Paul Painlevé [41] and the Swedish opthalmologist and Nobel laureate Allvar Gullstrand [42], who used them to criticize general relativity for allowing singularities to come and go! Their utility for studies of black hole quantum mechanics was emphasized more recently in [43].

For our purposes, the crucial features of these coordinates are that they are stationary and nonsingular through the horizon. Thus it is possible to define an effective "vacuum" state of a quantum field by requiring that it annihilate modes which carry negative frequency with respect to t; such a state will look essentially empty (in any case, nonsingular) to a freely-falling observer as he or she passes through the horizon. This vacuum differs strictly from the standard Unruh vacuum, defined by requiring positive frequency with respect to the Kruskal coordinate $U =$

$-\sqrt{r-2M}\exp\left(-\frac{t_s-r}{4M}\right)$ [44]. The difference, however, shows up only in transients, and does not affect the late-time radiation.

The radial null geodesics are given by

$$\dot{r} \equiv \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r}}\ , \tag{3.14}$$

with the upper (lower) sign in Eq. (3.14) corresponding to outgoing (ingoing) geodesics, under the implicit assumption that t increases towards the future. These equations are modified when the particle's self-gravitation is taken into account. Self-gravitating shells in Hamiltonian gravity were studied by Kraus and Wilczek [45]. They found that, when the black hole mass is held fixed and the total ADM mass allowed to vary, a shell of energy ω moves in the geodesics of a spacetime with M replaced by $M+\omega$. If instead we fix the total mass and allow the hole mass to fluctuate, then the shell of energy ω travels on the geodesics given by the line element

$$ds^2 = -\left(1 - \frac{2(M-\omega)}{r}\right)dt^2 + 2\sqrt{\frac{2(M-\omega)}{r}}\,dt\,dr + dr^2 + r^2 d\Omega^2\ , \tag{3.15}$$

so we should use Eq. (3.14) with $M \to M-\omega$.

Now one might worry that, since the typical wavelength of the radiation is of the order of the size of the black hole, a point particle description might be inappropriate. However, when the outgoing wave is traced back towards the horizon, its wavelength, as measured by local fiducial observers, is ever-increasingly blue-shifted. Near the horizon, the radial wavenumber approaches infinity and the point particle, or WKB, approximation becomes in fact excellent.

The imaginary part of the action for an s-wave outgoing positive energy particle which crosses the horizon outwards from r_{in} to r_{out} can be expressed as

$$\text{Im } S = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} p_r \, dr = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^{p_r} dp'_r \, dr \; . \tag{3.16}$$

Remarkably, this can be evaluated without entering into the details of the solution, as follows. We multiply and divide the integrand by the two sides of Hamilton's equation $\dot{r} = +\left.\frac{dH}{dp_r}\right|_r$, change variable from momentum to energy, and switch the order of integration to obtain

$$\text{Im } S = \text{Im} \int_0^{+\omega} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{dr}{1 - \sqrt{\frac{2(M-\omega')}{r}}} \, (-d\omega') \; , \tag{3.17}$$

where the minus sign appears because $H = M - \omega'$. But now the integral can be done by deforming the contour, so as to ensure that positive energy solutions decay in time (that is, into the lower half ω' plane). In this way we obtain

$$\text{Im } S = +4\pi\omega \left(M - \frac{\omega}{2} \right) \; , \tag{3.18}$$

provided $r_{\text{in}} > r_{\text{out}}$. To understand this ordering – which supplies the correct sign – we observe that when the integrals in Eq. (3.16) are not interchanged, and with the contour evaluated via the prescription $\omega \to \omega - i\epsilon$, we have

$$\text{Im } S = +\text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_M^{M-\omega} \frac{dM'}{1 - \sqrt{\frac{2M'}{r}}} \, dr = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} -\pi r \, dr \; . \tag{3.19}$$

Hence $r_{\text{in}} = 2M$ and $r_{\text{out}} = 2(M - \omega)$. (Incidentally, comparing the above equation with Eq. (3.16), we also find that Im $p_r = -\pi r$.) Thus, over the course of the classically forbidden trajectory, the outgoing particle travels radially inward with the apparent horizon to materialize at the *final* location of the horizon, viz. $r = 2(M - \omega)$.

Alternatively, and along the same lines, Hawking radiation can also be regarded as pair creation *outside* the horizon, with the negative energy particle tunneling into the black hole. Since such a particle propagates backwards in time, we have to reverse time in the equations of motion. From the line element, Eq. (3.13), we see that time-reversal corresponds to $\sqrt{\frac{2M}{r}} \to -\sqrt{\frac{2M}{r}}$. Also, since the anti-particle sees a geometry of fixed black hole mass, the upshot of self-gravitation is to replace M by $M+\omega$, rather than $M-\omega$. Thus an ingoing negative energy particle has

$$\mathrm{Im}\ S = \mathrm{Im} \int_0^{-\omega} \int_{r_{\mathrm{out}}}^{r_{\mathrm{in}}} \frac{dr}{-1+\sqrt{\frac{2(M+\omega')}{r}}}\, d\omega' = +4\pi\omega \left(M - \frac{\omega}{2}\right) , \tag{3.20}$$

where to obtain the last equation we have used Feynman's "hole theory" deformation of the contour: $\omega' \to \omega' + i\epsilon$.

Both channels – particle or anti-particle tunneling – contribute to the rate for the Hawking process so, in a more detailed calculation, one would have to add their amplitudes before squaring in order to obtain the semi-classical tunneling rate. That, however, only affects the pre-factor. In either treatment, the exponential part of the semi-classical emission rate, in agreement with [46], is

$$\Gamma \sim e^{-2\,\mathrm{Im}\ S} = e^{-8\pi\omega\left(M-\frac{\omega}{2}\right)} = e^{+\Delta S_{\mathrm{B-H}}} , \tag{3.21}$$

where we have expressed the result more naturally in terms of the change in the hole's Bekenstein-Hawking entropy, $S_{\mathrm{B-H}}$. When the quadratic term is neglected, Eq. (3.21) reduces to a Boltzmann factor for a particle with energy ω at the inverse Hawking temperature $8\pi M$. The ω^2 correction arises from the physics of energy conservation, which (roughly speaking) self-consistently raises the effective temperature of the hole as it radiates. That the exact result must be correct can be seen on physical grounds by considering the limit in which the emitted particle carries away the entire mass and charge of the black hole (corresponding to the transmutation of the black hole into an outgoing shell). There can be only one such outgoing state.

On the other hand, there are $\exp\left(S_{\mathrm{B-H}}\right)$ states in total. Statistical mechanics then asserts that the probability of finding a shell containing all the mass of the black hole is proportional to $\exp\left(-S_{\mathrm{B-H}}\right)$, as above.

Following standard arguments, Eq. (3.21) with the quadratic term neglected implies the Planck spectral flux appropriate to an inverse temperature of $8\pi M$:

$$\rho\left(\omega\right) = \frac{d\omega}{2\pi} \frac{\mid T\left(\omega\right)\mid^2}{e^{+8\pi M\omega} - 1} \,, \tag{3.22}$$

where $\mid T\left(\omega\right)\mid^2$ is the frequency-dependent (greybody) transmission co-efficient for the outgoing particle to reach future infinity without back-scattering. It arises from a more complete treatment of the modes, whose semi-classical behavior near the turning point we have been discussing.

3.5.1 Tunneling from a Charged Black Hole

When the outgoing radiation carries away the black hole's charge, the calculations are complicated by the fact that the trajectories are now also subject to electromagnetic forces. Here we restrict ourselves to uncharged radiation coming from a Reissner-Nordström black hole. The calculation then proceeds in an exactly similar fashion as in the case of Schwarzschild holes but, for completeness, we shall run through the corresponding equations.

The charged counterpart to the Painlevé line element is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + 2\sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}\, dt\, dr + dr^2 + r^2 d\Omega^2 \,, \tag{3.23}$$

which is obtained from the standard line element by the rather unedifying coordinate transformation,

$$t \;\; = \;\; t_r + 2\sqrt{2Mr - Q^2} + M \ln\left(\frac{r - \sqrt{2Mr - Q^2}}{r + \sqrt{2Mr - Q^2}}\right)$$

$$+\frac{Q^2-M^2}{\sqrt{M^2-Q^2}}\,\mathrm{arctanh}\left(\frac{\sqrt{M^2-Q^2}\sqrt{2Mr-Q^2}}{Mr}\right)\ , \qquad (3.24)$$

where t_r is the Reissner time coordinate. The equation of motion for an outgoing massless particle is

$$\dot{r} \equiv \frac{dr}{dt} = +1 - \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}\ , \qquad (3.25)$$

with $M \to M - \omega$ when self-gravitation is included. The imaginary part of the action for a positive energy outgoing particle is

$$\mathrm{Im}\ S = \mathrm{Im}\int_0^{+\omega}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \frac{dr}{1-\sqrt{\frac{2(M-\omega')}{r}-\frac{Q^2}{r^2}}}\,(-d\omega')\ , \qquad (3.26)$$

which is again evaluated by deforming the contour in accordance with Feynman's $w' \to w' - i\epsilon$ prescription. The residue at the pole can be read off by substituting $u \equiv \sqrt{2\,(M-\omega')\,r - Q^2}$. Finally, the emission rate is

$$\Gamma \sim e^{-2\,\mathrm{Im}\ S} = e^{-4\pi\left(2\omega\left(M-\frac{\omega}{2}\right)-(M-\omega)\sqrt{(M-\omega)^2-Q^2}+M\sqrt{M^2-Q^2}\right)} = e^{+\Delta S_{\mathrm{B-H}}}\ . \qquad (3.27)$$

To first order in ω, this is consistent with Hawking's result of thermal emission at the Hawking temperature, T_H, for a charged black hole:

$$T_H = \frac{1}{2\pi}\frac{\sqrt{M^2-Q^2}}{\left(M+\sqrt{M^2-Q^2}\right)^2}\ . \qquad (3.28)$$

But again, energy conservation implies that the exact result has corrections of higher order in ω; these can all be collected to express the emission rate as the exponent of the change in entropy.

We conclude this section by noting that only local physics has gone into our derivations. There was neither an appeal to Euclidean gravity nor a need to invoke an explicit collapse phase. The time asymmetry leading to outgoing radiation arose instead from use of the "normal" local contour deformation prescription in terms of the nonstatic coordinate t.

3.6 Emissions from the Membrane

As we have seen in the previous chapter, an outside observer can adopt another, rather different, description of a black hole, regarding the horizon effectively as a membrane at an inner boundary of spacetime. But again, quantum uncertainty in the position of the membrane complicates the issue in an essential way.

Before presenting the calculation, we briefly outline the strategy here. Starting with a fundamental action for the bulk, we obtain the membrane action as well as the bulk equations of motion. After continuing the equations to complex time, we look for solutions that connect the Lorentzian geometries before and after emission. Finally, we evaluate the action for our instanton to obtain the semi-classical rate.

The external action for an uncharged massless scalar field minimally coupled to gravity is

$$S[\varphi, g_{ab}] = +\frac{1}{16\pi}\int d^4x\sqrt{-g}\,R - \frac{1}{2}\int d^4x\sqrt{-g}\;(\partial\varphi)^2 + S_{\partial M} + S_{\text{surf}}\;. \tag{3.29}$$

Here the bulk terms have support only outside the black hole, and $S_{\partial M}$ is the term at the external boundaries of the spacetime manifold that is needed to obtain Einstein's equations [40]. The membrane action takes the classical value,

$$S_{\text{surf}}[\varphi, h_{ab}] = +\frac{1}{8\pi}\int d^3x\sqrt{-h}\,K - \int d^3x\sqrt{-h}\,\varphi J_s\;, \tag{3.30}$$

where h_{ab} is the metric induced on the stretched horizon, h is its determinant, and $K \equiv +\nabla_a n^a$ is the trace of the membrane's extrinsic curvature. The scalar field source, J_s, induced on the membrane is

$$J_s = +n^a\nabla_a\varphi\;, \tag{3.31}$$

with n^a the outward-pointing space-like normal to the membrane.

The field equations are Einstein's equations and the source-free Klein-Gordon equation. The energy-momentum tensor is

$$T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}\nabla_c\varphi\nabla^c\varphi \ , \tag{3.32}$$

whose trace is simply

$$T = -\left(\nabla\varphi\right)^2 \ . \tag{3.33}$$

From this we see that the Einstein-Hilbert and Klein-Gordon bulk actions cancel, so surface actions account for all the classical and semi-classical physics.

Next, we seek an instanton solution that connects the Lorentzian Schwarzschild geometry of mass M with a horizon at $r_H = 2M$, to a similar geometry of mass $M-\omega$ and a horizon at $r_H = 2\left(M-\omega\right)$. At this point, it is important to distinguish the stretched horizon, a surrogate for the globally-defined and acausal true horizon, from the locally-defined apparent horizon. For an evaporating hole, the true horizon (hence the stretched horizon) lies *inside* the apparent horizon; this is because the acausal true horizon shrinks in anticipation of future emissions before the local geometry actually changes. Hence, our analytically-continued solution must describe the geometry *interior* to the apparent horizon.

Now, in the usual analytic continuation ($t = -i\tau$), the apparent horizon is at the origin, and $r < 2M$ is absent from the Euclidean section of the geometry. However, there is no real need for a Euclidean section. Euclidean solutions may have positive Euclidean action, but the (Lorentzian) action for a general tunneling motion in a time-dependent setting need not be purely imaginary, and the instanton can be a more complicated trajectory in the complex time plane. It should not be surprising then that for a time-dependent shrinking black hole, one has to consider intermediate metrics of arbitrary signature. Indeed, the usual analytic continuation prescription yields, for $r < 2M$,

$$ds^2 = -\left(\frac{2M}{r} - 1\right) d\tau^2 - \left(\frac{2M}{r} - 1\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3.34}$$

which has two time-like coordinates. Substituting $x \equiv 4M\left(\frac{2M}{r} - 1\right)^{1/2}$, we eliminate the coordinate singularity at $r = 2M$ to obtain a spacetime with topology $S^1 \times S^2 \times R$, in which τ has a period of $8\pi M$ about the apparent horizon. The line element now describes the complexified geometry interior to the apparent horizon; moreover, it is regular everywhere except at the real singularity at $r = 0$.

In addition, we note that Birkhoff's theorem – spherically symmetric vacuum solutions are stationary – is valid irrespective of the metric signature. Hence, one Schwarzschild solution can go to another only if there is matter present. The matter is produced by the membrane during the complex time process, and materializes as the shell. We expect the membrane to be able to produce matter since it is also able to destroy (dissipate) matter; Hawking emission is the counterpart of absorption.

As matter is emitted during the emission process, the membrane is subject to a changing interior geometry which in turn implies a changing periodicity. In order to adapt Eq. (3.34) to a geometry with changing periodicity, we guess that the line element takes the form

$$ds^2 = -u(\psi)x^2 d\psi^2 - v(\psi, x)dx^2 + r^2(\psi, x)d\Omega^2 \tag{3.35}$$

which has a radial (x) and an angular (ψ) coordinate, both with dimensions of length. The period changes if $u(\psi)$, and $v(\psi, x)$ are not constant functions of ψ. The shell and the apparent horizon are at $x = 0$; the shell trajectory is parametrized by ψ. A convenient choice of ψ is one for which the shell's radius decreases linearly with ψ. Then the radius of the apparent horizon is

$$r_H = 2(M - b\psi)\ , \tag{3.36}$$

where b is dimensionless.

The important Christoffel symbols are

$$\Gamma^{\psi}_{\psi\psi} = +\frac{\dot{u}}{2u} \ , \quad \Gamma^{\psi}_{xx} = -\frac{\dot{v}}{2ux^2} \ , \quad \Gamma^{\psi}_{\psi x} = \frac{1}{x} \tag{3.37}$$

$$\Gamma^{x}_{\psi\psi} = -\frac{ux}{v} \ , \quad \Gamma^{x}_{xx} = +\frac{v'}{2v} \ , \quad \Gamma^{x}_{\psi x} = \frac{\dot{v}}{2v} \ , \tag{3.38}$$

where $\dot{}$ and $'$ denote differentiation by ψ and x, respectively. A normalized trajectory of constant x has

$$U^2 = -1 \ , \quad U^{\psi} = \frac{1}{x\sqrt{u}} \ , \tag{3.39}$$

so its proper acceleration has magnitude

$$\alpha = \frac{1}{x\sqrt{v}} \ . \tag{3.40}$$

The normal vector, n^a, normal to U^a obeys

$$n^2 = -1 \ , \quad n^x = \frac{1}{\sqrt{v}} \ . \tag{3.41}$$

Hence the trace of the extrinsic curvature of a surface of constant x is

$$K = \nabla_a n^a = \frac{1}{\sqrt{v}} \left(\frac{1}{x} + \frac{2}{r} r' \right) \ . \tag{3.42}$$

We will also need one component of the Ricci tensor:

$$R_{\psi x} = \frac{1}{r} \left(-2\dot{r}' + \frac{2}{x} \dot{r} + r' \frac{\dot{v}}{v} \right) \ . \tag{3.43}$$

Now the flux of energy-at-infinity (per local proper time), $\mathcal{F}$, is

$$\mathcal{F} = \frac{1}{4\pi r^2} \frac{d\omega'}{d\tau} = -\frac{1}{4\pi r^2} \frac{\dot{r}}{2} \frac{1}{x\sqrt{u}} \ . \tag{3.44}$$

This is related to the local stress tensor by $\mathcal{F} = -\sqrt{u} x T_{ab} U^a n^b$. Hence

$$T_{\psi x} = \frac{1}{2} \frac{1}{4\pi r^2} \sqrt{\frac{v}{u}} \frac{\dot{r}}{x} \ . \tag{3.45}$$

Comparing Eq. (3.45) with Eq. (3.43) in the $x \to 0$ limit, we find

$$\sqrt{\frac{u}{v}} = \frac{1}{2r} \, . \tag{3.46}$$

Incidentally, by Eq. (3.40), this implies that the "temperature-at-infinity" is

$$T_\infty = x\sqrt{u}\frac{\alpha}{2\pi} = \frac{1}{4\pi r_H} \, , \tag{3.47}$$

indicating a changing periodicity; as r_H decreases from $2M$ to $2(M-\omega)$, the temperature varies accordingly.

Now, for single-particle emission, both the true and the apparent horizon start out at $r = 2M$ and finish at $r = 2(M-\omega)$. Thus, the membrane moves along the apparent horizon. But now this is mapped to the origin by Euclideanization, so the stretched horizon has a vanishing proper volume element, even in Euclidean space. Thus, in the absence of a compensating divergence in the integrand, membrane integrals are zero. In particular, the scalar field current induced on the membrane has no divergence so the scalar part of the membrane action vanishes. Therefore the entire contribution to the emission rate comes from the gravitational term.

To evaluate that, note that there is a factor of x in the action, contained in $\sqrt{-g_{\psi\psi}}$. Then, as $x \to 0$,

$$xK \to \frac{1}{\sqrt{v}} \, . \tag{3.48}$$

Combining this with Eqs. (3.36) and (3.46), we have

$$S = \frac{1}{8\pi}\int d^3x\sqrt{-h}\,K = -\frac{i}{2}\int r^2\sqrt{\frac{u}{v}}d\psi = -\frac{i}{16b}\Delta r^2 \, . \tag{3.49}$$

When $b = \frac{1}{8\pi}$, we obtain the desired result, Eq. (3.18). This may be fixed by matching it with the rate $\exp(-4\pi M)$ for emission of the entire mass of the hole. Alternatively, we note that for single-particle emission from a Schwarzschild hole, there is no scale other than the mass of the hole. Since the instanton simply scales the horizon radius, we must have $K_H = g_H = \frac{1}{4M}$ throughout the motion. Hence the proper length along the stretched horizon, $d\tau \equiv \sqrt{u}d\psi$, must also scale. Thus, we have

$$\beta = \int d\tau = 8\pi M \Rightarrow d\tau = 8\pi dM \ , \tag{3.50}$$

so that $b = \frac{1}{8\pi}$, as required. We note here that b can be written as $\frac{M}{\beta}$ where β is the inverse temperature. As we shall see, with this form as an ansatz, the membrane action gives the right rate for emission from Reissner holes.

3.6.1 Emission from a Charged Membrane

For emission from charged black holes, there are a few modifications to the preceding equations. The action now has an additional term because of the electromagnetic field:

$$S[g_{ab}, A_a, \varphi] = +\frac{1}{16\pi}\int d^4x\sqrt{-g}\,R - \frac{1}{2}\int d^4x\sqrt{-g}\,(\partial\varphi)^2 - \frac{1}{16\pi}\int d^4x\sqrt{-g}\,F^2 + S_{\mathrm{surf}} \ , \tag{3.51}$$

which yields a membrane action of

$$S_{\mathrm{surf}}[h_{ab}, A_a, \varphi] = +\frac{1}{8\pi}\int d^3x\sqrt{-h}\,K - \int d^3x\sqrt{-h}\,\varphi J_s + \int d^3x\sqrt{-h}\,j_s^a A_a \ . \tag{3.52}$$

In addition to the scalar term induced on the membrane, there is now also an electromagnetic current, Eq. (2.16),

$$j_s^a = +\frac{1}{4\pi}F^{ab}n_b \ , \tag{3.53}$$

as we saw in the previous chapter. The stress tensor is

$$T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}\nabla_c\varphi\nabla^c\varphi + \frac{1}{4\pi}\left(F_{ac}F_b^{\ c} - \frac{1}{4}g_{ab}F^2\right) \ , \tag{3.54}$$

but the trace is still given by Eq. (3.33), so the Einstein-Hilbert and Klein-Gordon bulk actions again cancel. Now, in the absence of sources, the Maxwell action can be expressed as a surface integral:

$$-\frac{1}{16\pi}\int d^4x\sqrt{-g}\,F^2 = +\frac{1}{8\pi}\oint d^3x\sqrt{-h}\,n_a F^{ab} A_b\ , \tag{3.55}$$

where the sign on the right-hand side stems from choosing the normal to point into the bulk. This term is added to the membrane term to give a total electromagnetic action of

$$S^{\rm EM}_{\rm surf} = +\frac{1}{8\pi}\int d^3x\sqrt{-h}\,F^{ab} A_a n_b\ . \tag{3.56}$$

Thus, we have succeeded in eliminating all the bulk terms in the action. To evaluate the surface terms, we note first of all that, because the volume element along the membrane vanishes, the scalar membrane term contributes nothing to the rate. The electromagnetic and gravitational membrane terms can be combined, in the $x \to 0$ limit, to give

$$S_{\rm surf} = -\frac{i}{8\pi}\int d\psi\, 4\pi r^2 \left[\frac{\sqrt{M^2-Q^2}}{r^2} + \frac{Q^2}{r^3}\right]\ , \tag{3.57}$$

where we have skipped the steps analogous to Eqs. (3.36) - (3.46). Using

$$r_H = M - b\psi + \sqrt{(M-b\psi)^2 - Q^2}\ , \tag{3.58}$$

and our ansatz,

$$b \equiv \frac{M}{\beta} = \frac{M}{2\pi}\frac{\sqrt{M^2-Q^2}}{r^2}\ , \tag{3.59}$$

we have that

$$S_{\rm surf} = -i\frac{\pi}{2}\Delta r_+^2\ , \tag{3.60}$$

which yields the correct tunneling rate.

When the hole emits charged radiation, the analysis become more complicated. However, one can consider the case in which the emitted radiation has the same charge-to-mass ratio as the hole itself. Then the problem again becomes one of scaling. Letting $Q \equiv \eta M$ with $|\eta| < 1$, we have

$$\beta = 2\pi \frac{\left(1+\sqrt{1-\eta^2}\right)^2}{\sqrt{1-\eta^2}} M \Rightarrow d\tau = 2\pi \frac{\left(1+\sqrt{1-\eta^2}\right)^2}{\sqrt{1-\eta^2}} dM \ . \tag{3.61}$$

With $K_H = g_H$, we have

$$S = -i \int \pi M \left(1+\sqrt{1-\eta^2}\right)^2 dM = -i\frac{\pi}{2}\Delta r_+^2 \ . \tag{3.62}$$

Calling the change in the hole's charge q, the emission rate to first order in ω and q is Hawking's result,

$$\Gamma \sim e^{-\frac{2\pi}{g_H}(\omega - q\Phi)} \ , \tag{3.63}$$

where $\Phi \equiv +Q/r_+$ is the electromagnetic scalar potential at the horizon; emissions which discharge the membrane are favored. The exact rate is again proportional to the exponent of the change in entropy.

Chapter 4

The Causal Structure Of Evaporating Black Holes

4.1 Introduction

It is challenging to envision a plausible global structure for a spacetime containing a decaying black hole. If information is not lost in the process of black hole decay, then the final state must be uniquely determined by the initial state, and vice versa. Thus a post-evaporation space-like hypersurface must lie within the future domain of dependence of a pre-evaporation Cauchy surface. One would like to have models with this property that support approximate (apparent) horizons.

In addition, within the framework of general relativity, one expects that singularities will form inside black holes [47]. If the singularities are time-like, one can imagine that they will go over into the world-lines of additional degrees of freedom occurring in a quantum theory of gravity. Ignorance of the nature of these degrees of freedom is reflected in the need to apply boundary conditions at such singularities. (On the other hand, boundary conditions at future space-like singularities represent constraints on the initial conditions; it is not obvious how a more complete dynamical theory could replace them with something more natural.)

In this chapter, we use the charged Vaidya metric to obtain a candidate macroscopic Penrose diagram for the formation and subsequent evaporation of a charged black hole, thereby illustrating how predictability might be retained. We do this by first extending the charged Vaidya metric past its coordinate singularities, and then joining together patches of spacetime that describe different stages of the evolution.

4.2 Extending the Charged Vaidya Metric

The Vaidya metric [48] and its charged generalization [49, 50] describe the spacetime geometry of unpolarized radiation, represented by a null fluid, emerging from a spherically symmetric source. In most applications, the physical relevance of the Vaidya metric is limited to the spacetime outside a star, with a different metric describing the star's internal structure. But black hole radiance [3] suggests use of the Vaidya metric to model back-reaction effects for evaporating black holes [51, 52] all the way upto the singularity.

The line element of the charged Vaidya solution is

$$ds^2 = -\left(1 - \frac{2M(u)}{r} + \frac{Q^2(u)}{r^2}\right) du^2 - 2\,du\,dr + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \tag{4.1}$$

The mass function $M(u)$ is the mass measured at future null infinity (the Bondi mass) and is in general a decreasing function of the outgoing null coordinate, u. Similarly, the function $Q(u)$ describes the charge, measured again at future null infinity. When $M(u)$ and $Q(u)$ are constant, the metric reduces to the stationary Reissner-Nordström metric. The corresponding stress tensor describes a purely electric Coulomb field,

$$F_{ru} = +\frac{Q(u)}{r^2} \ , \tag{4.2}$$

and a null fluid with current

$$k_a = k\nabla_a u \ , \quad k^2 = +\frac{1}{4\pi r^2}\frac{\partial}{\partial u}\left(-M + \frac{Q^2}{2r}\right) \ . \tag{4.3}$$

In particular,

$$T_{uu} = \frac{1}{8\pi r^2}\left[\left(1 - \frac{2M(u)}{r} + \frac{Q^2(u)}{r^2}\right)\frac{Q^2(u)}{r^2} + \frac{1}{r}\frac{\partial Q^2(u)}{\partial u} - 2\frac{\partial M(u)}{\partial u}\right] \ . \tag{4.4}$$

Like the Reissner-Nordström metric, the charged Vaidya metric is beset by coordinate singularities. It is not known how to remove these spurious singularities for arbitrary mass and charge functions (for example, see [53]). We shall simply choose functions for which the relevant integrations can be done and continuation past the spurious singularities can be carried out, expecting that the qualitative structure we find is robust.

Specifically, we choose the mass to be a decreasing linear function of u, and the charge to be proportional to the mass:

$$M(u) \equiv au + b \equiv \tilde{u} \ , \quad Q(u) \equiv \eta\tilde{u} \ , \tag{4.5}$$

where $a < 0$ and $|\eta| \le 1$, with $|\eta| = 1$ at extremality. We always have $\tilde{u} \ge 0$. With these choices, we can find an ingoing (advanced time) null coordinate, v, with which the line element can be written in a "double-null" form:

$$ds^2 = -\frac{g(\tilde{u}, r)}{a} d\tilde{u}\, dv + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \tag{4.6}$$

Thus

$$dv = \frac{1}{g(\tilde{u}, r)}\left[\left(1 - \frac{2\tilde{u}}{r} + \frac{\eta^2\tilde{u}^2}{r^2}\right)\frac{d\tilde{u}}{a} + 2\, dr\right] \ . \tag{4.7}$$

The term in brackets is of the form $X(\tilde{u},r)\,d\tilde{u} + Y(\tilde{u},r)\,dr$. Since $X(\tilde{u},r)$ and $Y(\tilde{u},r)$ are both homogeneous functions, Euler's relation provides the integrating factor: $g(\tilde{u},r) = X(\tilde{u},r)\tilde{u} + Y(\tilde{u},r)r$. Hence

$$\frac{\partial v}{\partial r} = \frac{r^2}{r^3 + \frac{\tilde{u}}{2a}(r^2 - 2\tilde{u}r + \eta^2\tilde{u}^2)} \tag{4.8}$$

$$\frac{\partial v}{\partial \tilde{u}} = \frac{\frac{1}{2a}(r^2 - 2\tilde{u}r + \eta^2\tilde{u}^2)}{r^3 + \frac{\tilde{u}}{2a}(r^2 - 2\tilde{u}r + \eta^2\tilde{u}^2)} \, . \tag{4.9}$$

From the sign of the constant term of the cubic, we know that there is at least one positive zero. Then, calling the largest positive zero r', we may factorize the cubic as $(r-r')(r^2+\beta r+\gamma)$. Hence

$$\gamma = -\frac{\eta^2\tilde{u}^3}{2ar'} > 0 \; , \quad \gamma - \beta r' = -\frac{\tilde{u}^2}{2a} > 0 \; , \quad \beta - r' = \frac{\tilde{u}}{2a} < 0 \, . \tag{4.10}$$

Consequently, the cubic can have either three positive roots, with possibly a double root but not a triple root, or one positive and two complex (conjugate) roots. We consider these in turn.

i) Three positive roots

When there are three distinct positive roots, the solution to Eq. (4.8) is

$$v = A\ln(r-r') + B\ln(r-r_2) + C\ln(r-r_1) \; , \tag{4.11}$$

where $r' > r_2 > r_1 > 0$, and

$$A = \frac{+r'^2}{(r'-r_2)(r'-r_1)} > 0 \; , \quad B = \frac{-{r_2}^2}{(r'-r_2)(r_2-r_1)} < 0 \; , \quad C = \frac{+{r_1}^2}{(r'-r_1)(r_2-r_1)} > 0 \, . \tag{4.12}$$

We can push through the r' singularity by defining a new coordinate,

$$V_2(v) \equiv e^{v/A} = (r-r')(r-r_2)^{B/A}(r-r_1)^{C/A} \; , \tag{4.13}$$

which is regular for $r > r_2$. To extend the coordinates beyond r_2 we define

$$V_1(v) \equiv k_2 + (-V_2)^{A/B} = k_2 + (r' - r)^{A/B}(r_2 - r)(r - r_1)^{C/B} \ , \tag{4.14}$$

where k_2 is some constant chosen to match V_2 and V_1 at some $r' > r > r_2$. $V_1(r)$ is now regular for $r_2 > r > r_1$. Finally, we define yet another coordinate,

$$V(v) \equiv k_1 + (-(V_1 - k_2))^{B/C} = k_1 + (r' - r)^{A/C}(r_2 - r)^{B/C}(r - r_1) \ , \tag{4.15}$$

which is now free of coordinate singularities for $r < r_2$. A similar procedure can be applied if the cubic has a double root.

ii) One positive root

When there is only one positive root, v is singular only at $r = r'$:

$$v = A \ln(r - r') + \frac{1}{2} B \ln(r^2 + \beta r + \gamma) + \frac{2C - B\beta}{\sqrt{4\gamma - \beta^2}} \arctan\left(\frac{2r + \beta}{\sqrt{4\gamma - \beta^2}}\right) \ . \tag{4.16}$$

We can eliminate this coordinate singularity by introducing a new coordinate

$$V(v) \equiv e^{v/A} = (r - r')(r^2 + \beta r + \gamma)^{B/2A} \exp\left[+\frac{2C - B\beta}{A\sqrt{4\gamma - \beta^2}} \arctan\left(\frac{2r + \beta}{\sqrt{4\gamma - \beta^2}}\right)\right] \ , \tag{4.17}$$

which is well-behaved everywhere. The metric now reads

$$ds^2 = -g(\tilde{u}, r)\frac{A}{V(\tilde{u}, r)} \frac{d\tilde{u}}{a} dV + r^2 d\Omega^2 \ . \tag{4.18}$$

In all cases, to determine the causal structure of the curvature singularity we express dV in terms of du with r held constant. Now we note that, since $\tilde{u}$ is the only dimensionful parameter, all derived dimensionful constants such as r' must be proportional to powers of $\tilde{u}$. For example, when there is only positive zero, Eq. (4.17) yields

$$dV|_r = d\tilde{u}\, \frac{V}{\tilde{u}} \left[\frac{-r'}{r - r'} + \frac{B}{2A} \frac{\beta r + 2\gamma}{r^2 + \beta r + \gamma} + \frac{2C - B\beta}{A\sqrt{4\gamma - \beta^2}} \frac{1}{1 + \left(\frac{2r+\beta}{\sqrt{4\gamma - \beta^2}}\right)^2} \frac{-2r}{\sqrt{4\gamma - \beta^2}}\right] \ . \tag{4.19}$$

Thus, as $r \to 0$, and using the fact that $A + B = 1$, we have

$$ds^2 \to -\frac{Q^2(u)}{r^2}\, du^2 \ , \tag{4.20}$$

so that the curvature singularity is time-like.

4.3 Patches of Spacetime

Our working hypothesis is that the Vaidya spacetime, since it incorporates radiation from the shrinking black hole, offers a more realistic background than the static Reissner spacetime, where all back-reaction is ignored. In this spirit, we can model the black hole's evolution by joining patches of the collapse and post-evaporation (Minkowski) phases onto the Vaidya geometry.

To ensure that adjacent patches of spacetime match along their common boundaries, we can calculate the stress-tensor at their (light-like) junction. The absence of a stress-tensor intrinsic to the boundary indicates a smooth match when there is no explicit source there. Surface stress tensors are ordinarily computed by applying junction conditions relating discontinuities in the extrinsic curvature; the appropriate conditions for light-like shells were obtained in [54]. However, we can avoid computing most of the extrinsic curvature tensors by using the Vaidya metric to describe the geometry on both sides of a given boundary, because the Reissner-Nordström and Minkowski spacetimes are both special cases of the Vaidya solution.

Initially then, we have a collapsing charged spherically symmetric light-like shell. Inside the shell, region I, the metric must be that of flat Minkowski space; outside, region II, it must be the Reissner-Nordström metric, at least initially. In fact, we can describe both regions together by a time-reversed charged Vaidya metric,

$$ds^2 = -\left(1 - \frac{2M(v)}{r} + \frac{Q^2(v)}{r^2}\right) dv^2 + 2\,dv\,dr + r^2 d\Omega^2 \ , \tag{4.21}$$

where the mass and charge functions are step functions of the ingoing null coordinate:

$$M(v) = M_0 \Theta(v - v_0) \ , \quad Q(v) = \eta M(v) \ . \tag{4.22}$$

The surface stress tensor, t^s_{vv}, follows from Eq. (4.4). Thus

$$t^s_{vv} = \frac{1}{4\pi r^2}\left(M_0 - \frac{Q_0^2}{2r}\right) \ . \tag{4.23}$$

The shell, being light-like, is constrained to move at 45 degrees on a conformal diagram until it has collapsed completely. Inside the shell, the spacetime is guaranteed by Birkhoff's theorem to remain flat until the shell hits $r = 0$, at which point a singularity forms.

Meanwhile, outside the shell, we must have the Reissner-Nordström metric. This is appropriate for all $r > r_+$. Once the shell nears r_+, however, one expects that quantum effects start to play a role. For nonextremal ($|\eta| < 1$) shells, the Killing vector changes character – time-like to space-like – as the apparent horizon is traversed, outside the shell. This permits a virtual pair, created by a vacuum fluctuation just outside or just inside the apparent horizon, to materialize by having one member of the pair tunnel across the apparent horizon. Thus, Hawking radiation begins, and charge and energy will stream out from the black hole.

We shall model this patch of spacetime, region III, by the Vaidya metric. This must be attached to the Reissner metric, region II, infinitesimally outside $r = r_+$. A smooth match requires that there be no surface stress tensor intrinsic to the boundary of the two regions. The Reissner metric can be smoothly matched to the radiating solution along the $u = 0$ boundary if $b = M_0$ in Eq. (4.5).

Now, using Eqs. (4.7) and (4.17), one can write the Vaidya metric as

$$ds^2 = -\frac{g^2(\tilde{u}, r)A}{\left(1 - \frac{2M(u)}{r} + \frac{Q^2(u)}{r^2}\right) V^2} dV^2 + 2\frac{g(\tilde{u}, r)A}{\left(1 - \frac{2M(u)}{r} + \frac{Q^2(u)}{r^2}\right) V} dV\, dr \ . \qquad (4.24)$$

We shall assume for convenience that $g(r)$ has only one positive real root, which we call r'. Then, since V and g both contain a factor $(r - r')$, Eq. (4.17), the above line element and the coordinates are both well-defined for $r > r_+(\tilde{u})$. In particular, $r = \infty$ is part of the Vaidya spacetime patch. Moreover, the only solution with $ds^2 = dr = 0$ also has $dV = 0$, so that there are no light-like marginally trapped surfaces analogous to the Reissner $r_\pm$. In other words, the Vaidya metric extends to future null infinity, $\mathcal{I}^+$, and hence there is neither an event horizon, nor a second time-like singularity on the right of the conformal diagram.

The singularity on the left exists until the radiation stops, at which point one has to join the Vaidya solution to Minkowski space. This is easy: both spacetimes are at once encompassed by a Vaidya solution with mass and charge functions

$$M(u) = (au + b)\Theta(u_0 - u) \ , \quad Q(u) = \eta M(u) \ . \qquad (4.25)$$

As before, the stress tensor intrinsic to the boundary at u_0 can be read off Eq. (4.4):

$$t^s_{uu} = \frac{1}{4\pi r^2}\left[(au + b) - \frac{(au + b)^2}{2r}\right] \ , \qquad (4.26)$$

which is zero if $u_0 = -b/a$, i.e., if $\tilde{u} = 0$. This says simply that the black hole must have evaporated completely before one can return to flat space.

Collecting all the constraints from the preceding paragraphs, we can put together a possible conformal diagram, as in Fig. 1. (We say "possible" because a similar analysis for an uncharged hole leads to a space-like singularity; thus our analysis demonstrates the possibility, but not the inevitability, of the behavior displayed in

Fig. 1.) Fig. 1 is a Penrose diagram showing the global structure of a spacetime in which a charged imploding null shock wave collapses catastrophically to a point and subsequently evaporates completely. Here regions I and IV are flat Minkowski space, region II is the stationary Reissner-Nordström spacetime, and region III is our extended charged Vaidya solution. The zigzag line on the left represents the singularity, and the straight line separating region I from regions II and III is the shell. The curve connecting the start of the Hawking radiation to the end of the singularity is $r_+(\tilde{u})$, which can be thought of as a surface of pair creation. The part of region III interior to this line might perhaps be better approximated by an ingoing negative energy Vaidya metric.

From this cut-and-paste picture we see that, given some initial data set, only regions I and II and part of region III can be determined entirely; an outgoing ray starting at the bottom of the singularity marks the Cauchy horizon for these regions. Note also that there is no true horizon; the singularity is naked. However, because the singularity is time-like, Fig. 1 has the attractive feature that predictability for the entire spacetime is restored if conditions at the singularity are known. It is tempting to speculate that, with higher resolution, the time-like singularity might be resolvable into some dynamical Planck-scale object such as a D-brane.

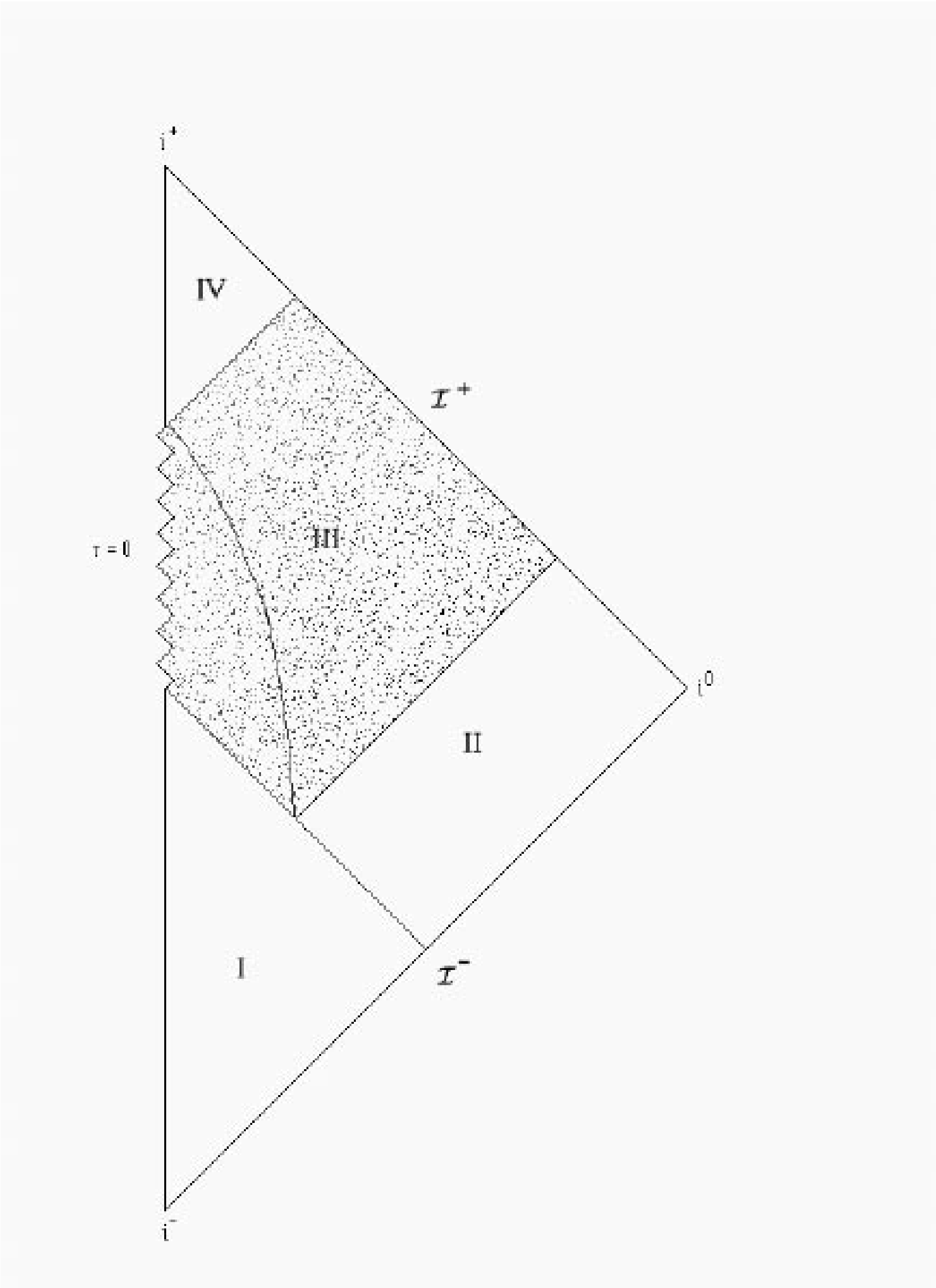

Figure 4.1: Penrose diagram for the formation and evaporation of a charged black hole.

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